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BOSTON UNIVERSITY

GRADUATE SCHOOL

Thesis

THE DEVELOPMENT OF APPRECIATION UNITS IN SECONDARY MATHEMATICS.

Submitted by

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(A.B., Boston University, 1921)

In partial fulfilment of requirements
for the degree of Master of Arts

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I wish to acknowledge indebtedness to Professors Robert E. Bruce and Norton A. Kent for contributions and information accuired in their classes and to Professor Guy M. Wilson for the helpful and stimulating suggestions he has given me in the selection and preparation of these appreciation units in secondary matheastics.

I. Introduction

A. The Appreciation Viewpoint.

B. Mathematics and the Progress of Science.

A. The Appreciation Viewpoint.

Ever since man came to think in the abstract, to create units of measure, to think of time, and to be aware of such concepts as lines and angles, mathematics has been the object of his study and the basis of the natural sciences of (1) antiquity, becoming the very essence of all science of the present day. In an age that seems to be more utilitarian than its immediate predecessors, it is natural, in weighing the value of mathematics, to consider its manifold uses. We study mathematics because it is one of a small group of subjects that are linked up with a large number of the branches of human knowledge. The pupil today, then, needs an intelligible conception of the immeasurable contribution mathematics has made to the comfort, convenience, and power of man. He must realize that, not only would the greatest of such achievements have been utterly impossible, except for mathematics, but that the less important developments which could have been wrought without mathematical help could not have been thus accomplished without tremendous waste of material and human energy. He must also understand that the field of application is at present widening at an extremely rapid rate. Finally, he must be convinced that, "if all the contributions which mathematics has made, and which nothing else could make, to navigation, to the building of railways, to the construction of ships, to the subjugation of wind and wave, electricity and heat, and many

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(1) Smith, D. E. - The Call of Mathematics - The Mathematics Teacher, May, 1929 - Vol. XIX
P. 282

other forms and manifestations of energy, he will have demonstrated that, if all these contributions of mathematics were suddenly withdrawn, the life and body of industry and commerce would suddenly collapse as by a paralytic stroke, the now splendid outer tokens of material civilization would perish and the face of our planet would quickly assume the aspect of a ruined and bankrupt world." (1)

Efficient learning does not require the mastery of the intricacies of the theories of mathematics. It does, however, go hand in hand with interest in the subject. The teacher's task, then, is to find problems that are gripping and that will stimulate to purposeful activity. In modern mathematics teaching the development of an appreciation view point is absolutely essential. To appreciate mathematics means to become aware of the sweep, the grandeur, and the universality of the science--and to venerate it. It means to instill a desire to read its histories, the biographies of its men of genius. It means a recognition of the truths discovered by it; the doctrines created by it; the influence of these, through their applications and their beauty, upon the advancement of civilization and the welfare of man.

To develop this appreciation view point certain units have been selected as appreciation units of secondary mathematics. These are not regular lessons and under no consideration is there to be a test on the material involved. These units are merely incidental information to

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(1) Keyser, C. J. - The Human Worth of Rigorous Thinking.
Columbia University Press - Page 10

show how mathematical knowledge has been of first importance in determining the nature of the universe in which we live and in carrying out many of the scientific and technological investigations that were essential in attaining the material prosperity and the comfort and luxury that are characteristic of the present age. Some of the newer applications are also considered to show the pupils that mathematics opens to their minds many doors which will in all likelihood forever remain closed, if in the high school, the opportunity to begin this study is neglected.

B. Mathematics and the Progress of Science.

The fundamental mathematical theory has always developed independently of the scientific phenomena which it explains. Again and again it would almost seem that the experimental scientists waited, to allow the mathematician to go ahead and pave the way. The astronomers, the physicists, all scientists who have made progress have approached the science, in general, with adequate mathematical equipment.

Take such simple curves as the conic sections associated through the analytical geometry with quadratic equations. These curves first as cut from cones with varying vertex angles, the ancient Greeks studied simply and solely because these curves seemed to them to follow upon the circle. The properties of these curves were carefully studied by the Greek mathematicians without any idea of making a practical application of them.

Nearly 2000 years later, Kepler, a student of mathematics and astronomy, found that the orbit of the earth was not circular. He turned naturally to the ellipse with whose properties, as a serious student of the mathematics of his day, he was familiar. So he was able to determine that the planets move in ellipses with the sun at one focus. Through his familiarity with the conic sections, he was able to enunciate two other laws with regard to the rate of motion and the distance of the planets from the sun.

Neither the Greeks who began these studies nor Kepler when he studied the Greek developments had any notion of their applications to astronomical problems. Had Kepler not had this background of mathematical information and training the world would have waited considerably longer before the astronomical facts would have received their correct interpretation.

Not long afterwards Galileo Galilei wished to investigate the motion of a falling body. He was able to show, by a simple mathematical law, that the space passed over in successive seconds was proportional to the successive odd numbers. As a result, the total space passed over by a freely falling body is given by the quadratic equation, $s=16\ t^2$.

During the latter part of the seventeenth century, Newton was able to verify his hypothesis as to the law of gravitation by showing that Kepler's three laws were consequences of this more fundamental law. He was able to do this by means of a new and powerful mathematical method which he had just invented, namely the calculus. Newton's great discovery, which would have been impossible without the use of the highest mathematical knowledge available at his time, represented the longest step that man had taken in the comprehension of the inner nature of our universe.

Today the automobile engineer uses the parabola, one of the conic sections, to fashion the automobile headlight; the architect uses the parabola to build the auditorium and to build his finest bridges; the student of pro-

jectiles begins with the parabola. Progress is made by the use of these geometric curves with simple algebraic properties. The teacher today who does not point out their many uses in practical affairs misses a great opportunity to impress upon the pupil the part of mathematics in the progress of science and to impress upon the pupil the universe as ruled by mathematical law.

The outstanding characteristic of a great physicist or astronomer is the ability to interpret and extend his observations by means of mathematical formulas. Only rarely is there an exception to this rule. Such an exception was doubtless Faraday whose contributions to the science of electricity press in upon us wherever we live. Faraday himself knew that lack of familiarity with mathematics was a serious drawback. Using Faraday's observations, Clerk Maxwell evolved the famous Maxwellian equations. From these mathematical investigations Maxwell concluded that electrical disturbances were propagated in waves. Somewhat later a German physicist, Hertz, verified experimentally the existence of these waves. Without the basis of this mathematical formulation the marvels of wireless telegraphy and radio could not have been achieved in our day and age.

About this same time, by representing points on a line symmetrically placed with respect to an origin, a great step was made for advance in the theory of equations. Soon the negative and irrational solutions of equations came to be accepted on a par with solutions

in rational numbers. The complete acceptance of the complex number waited upon some graphical representation. To this problem no less than three mathematicians, the Norwegian Kaspar Wessell, the great German scholar Gauss, and the Frenchman Argand found independently the solution. The result is the complex number diagram which is known today even to our high school students. This conception gave to that great genius Gauss the suggestion for the proof of the so-called fundamental theorem of algebra, the theorem that every polynomial in "x" with real and complex coefficients has a root.

It was not long, however, before the scientist discovered that this new speculation of the mathematician afforded simplicity in the consideration of his own problems. Thus Steinmetz, employing the symbolism and the methods of operation with complex numbers, made the theory of alternating currents easily intelligible. Here, then, is another illustration that progress in science has been preceded by progress in mathematics.

There can be no doubt, so far as the science of physics and astronomy is concerned, that progress has been expressed in mathematical formulas for the laws of physics and the laws of planetary motion are simple mathematical statements. If we turn now from the study of the past to the consideration of the present and the future, we shall soon see that the opportunities for applying mathematical knowledge are more numerous and

varied than at any previous time and that their number and variety are rapidly increasing. Today the chemist, the biologist, and even the student of medicine, are looking to the mathematician in an attempt to formulate the phenomena of the test tube, the growing plant and the human body. As the knowledge in such complex fields as meteorology, sociology, economics, psychology, and education is continually gaining in precision, it is obvious that the possibilities for applying mathematical methods must rapidly increase in number in the near future.

In the field of medicine one would least expect to find an application of higher mathematics. Before the outbreak of the late war Dr. Alexis Carrel, of the Rockefeller Institute of Medical Research, had noticed in the course of some of his experiments that the rate of healing a wound seemed to be approximately proportional to its surface area. Later, one of his colleagues, Dr. P Lecomte du Nouy, found that the area of the wound surface at any given time could be expressed in terms of the area at the time of first observation, the interval of time elapsed since that observation, and a certain quantity "i" known as the index of the wound. Once "i" was determined it was possible to plot a curve of healing and predict the future progress of the wound. Any marked departure from the curve was due to some abnormality such as infection in the wound—quite a valuable application of higher mathematics.

One of the most obvious of economic phenomena is the

alternation of periods of prosperity and periods of depressions. By simple mathematical analysis, the economist has established the cause of these economic cycles and the law which governs them. In meteorology and scientific agriculture the methods employed are the methods of mathematical statistics and the results are noteworthy. For example, in the domain of meteorology proper, the methods of mathematical statistics have enabled the meteorologists to establish curious and unexpected connections between weather happenings in various parts of the globe. It is evident that we are on our way to an exact science of meteorology, and it is by the use of mathematical methods that we shall arrive at that goal.

Can the high school student of today study physics and other science first and leave the necessary mathematical formulas and methods until the need arises? The secondary school teacher, fortified by this wealth of evidence, is well equipped to answer this vital question. The answer of history is absolutely and emphatically, No. Had Kepler not been familiar with the mathematical theory of the conic sections he would not have been able to formulate his laws of planetary motion. Had Newton not been acquainted with the mathematical advances of his day, his conception of the universe could not have been formed. When his work in medicine brought to his mind the problems of sound Helmholtz had the indispensable preparation and was able to solve problems which would have

been for him forever insolvable if he had been required to go back to build up the mathematical foundation. Maxwell, too, had that thorough mathematical preparation which enabled him to see the formulas back of the electromagnetic phenomena. A whole host of recent physicists like Michelson and (1) Einstein have had such a profound grasp of mathematics that the new physics, a mathematical creation has emerged. The mathematical training of these men began in their early student days and it has continued ever since.

Not one of these thinkers of the past ages felt that his search for mathematical truth needed the practical application to justify his study. Even in the ancient days, the circle, the parabola, and the ellipse were realities, and the numbers of arithmetic and the symbols of algebra, geometry, trigonometry and the calculus created a world whose reality can not now be disputed.

We have now considered in a brief way some of the past and present contributions of mathematical ideas to the progress of science and the advance of civilization. Surely one can not fail to be impressed by the fact that mathematics, the oldest of the sciences, is indeed the 'handmaid of the sciences'--always in advance, ever returning to help. It should be apparent then that the future citizen who engages in some form of intellectual activity is very apt to have need of mathematical knowledge and mathematical train-

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(1) Karpinski, L. - Mathematics and the Progress of Science
School Science and Mathematics
February, 1929 - page 131.

ing. For the boys and girls in high school the lack of instruction in elementary trigonometry, in elementary algebra, in geometry will absolutely bar their way to the sciences; it is as serious for them as lack of mathematical training is for the physicist today. In college it is too late to turn back to pick up that elementary instruction in the mathematics which today, more than ever, is essential for the progress of science. The student who waits to learn his mathematics and theory as he needs it, that student will never find it necessary. Progress will not wait for him.

Since the high school is obviously the place where the majority of our future citizens must obtain whatever serious training in mathematics, beyond arithmetic, they are apt to receive we must see that they get this training both for the sake of society and the sake of the individual. We cannot do less and do our duty as educators. It is for this reason--to stimulate interest, to create the will to learn--that certain appreciation units, worked out in detail, may be used to advantage in secondary school mathematics. Once impressed with the beauty, the grandeur, and the utility of the subject, the average pupil will desire to continue beyond the required course of secondary mathematics. By creating such a desire, secondary school teachers of today may feel assured that those pupils who in the future may need mathematics, will have received, from their own volition, the necessary background and training. Much of the material used in illustrating the power and beauty of mathematics cannot, of course, be made intelligible to our high

school pupils, but, nevertheless, they will feel the atmosphere of the teacher's interest and catch the contagion of her faith. They may not follow all the intricacies of the theories examined but the enthusiasm and evident genius involved can not fail to seize them with a grip that they shall never forget.

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APPRECIATION UNIT I

THE MATHEMATICS USED BY KEPLER IN FORMULATING THE LAWS OF PLANETARY MOTION

KEPLER'S LAWS OF PLANETARY MOTION

Kepler's chief claim to memory lies in his enunciation of the laws of planetary motion. The record of this faithful follower of science is remarkable. He displayed a genius and enthusiasm for mathematics and astronomy which led him to the highest pinnacles of scientific achievement.

Kepler was denied the joy of astronomical observation, for in his youth a serious illness had permanently injured his eyesight. He conquered, but his victory was won in the battlefield of geometry and statistics with the aid of his calculations, his drawing instruments, and above all his wonderful perseverance in the face of repeated failure.

We must not neglect to do homage to Kepler's great influence on his contemporaries in the realms of mathematics, and particularly in geometry. This branch of mathematics had suffered some neglect at the expense of a great advance in algebra, the theory of equations, and trigonometry. The seventeenth century revival in geometry was largely due to the influence of Kepler. It was he who introduced to the world of mathematics for the first time what has been called the "principle of continuity."

As an instance of this, he regarded the circle as a (1) special case of the ellipse. For, keeping the major axis AA', in Figure 1 below, of constant length, suppose we con-

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Hart, Ivor B. - Makers of Science - Oxford University Press
Page 85.

struct a series of ellipses with the two foci SS' (Kepler was the originator of the term 'focus') successively closer together, we find that the minor axis becomes longer and longer, making the ellipse resemble the circle more and more closely, until we reach the limiting case in which the foci coincide with the centre, when the curve becomes a circle.

Kepler's Principle of Continuity. The Circle as a Special Case of the Ellipse.

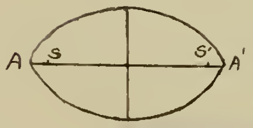


Fig. 1

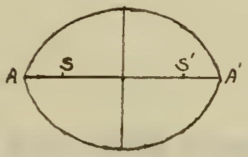


Fig. 2

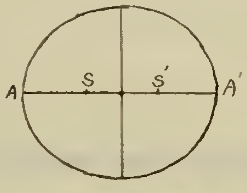


Fig. 3

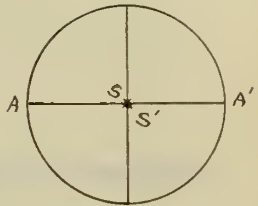
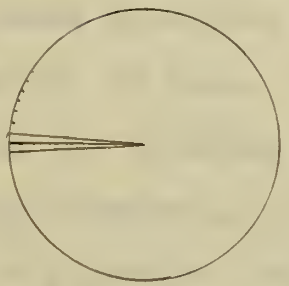


Fig. 4

We have another example of the principle of continuity when we regard the parabola as continuous with the ellipse. This is seen by keeping A and S (Figure 1) fixed, and gradually moving the other focus S' and the extremity A' farther and farther to the right. In the limit, when S' is at an infinite distance to the right, the curve becomes a parabola. Kepler introduced the idea of infinitesimals as an application of the law of continuity. This was really a simple extension of the method of limits in vogue as far back as the days of Euclid and Archimedes, but it had far-reaching consequences later, for it paved the way to the invention of the calculus by Newton and Leibnitz. By way of illustration of Kepler's introduction of the term "infinitely small quantity", a single case will suffice. He regarded

a circle as being made up of an infinite number of triangles all with vertices at the centre O, and with infinitely small bases on the circumference. The sum of the areas of the infinitely small triangles gives the total area of the circle, while the sum of the lengths of the infinitely small bases gives the length of the circumference.



Showing application of Infinitesimals to the Area of a Circle.

During all his life time, Kepler was deeply concerned with the six known planets--Mercury, Venus, Earth, Mars, Jupiter and Saturn. He knew that these were at successively greater distances from the sun. Moreover he knew that the farther a planet was from the sun, the slower its motion. It was Kepler's strong feeling that there was in all this some governing scheme. The unravelling of this problem he made his life work. His final efforts were indeed crowned with success, but his first theory was fantastic. An important outcome, however, was a cordial invitation which he now received from a man at Prague who was destined to affect his whole future career. This man was the famous Tycho Brahe, justly termed the pioneer of accurate astronomical observation. Tycho was not a follower of Copernicus but he nevertheless had sound advice to offer Kepler. "Do not build up abstract speculations concerning the system

of the world," he advised, "but rather first lay a solid foundation in observations, and then by ascending from them, strive to come at the causes of things." Kepler, of course, did not hesitate to abandon his fantastic theory when he realized it was not consistent with the accurate observations of Tycho Brahe.

About this time Kepler received an appointment to the University of Linz. At the same time he was applying his mind to the problems of the solar system, and one by one, at long intervals, he gave to the world his three wonderful laws of planetary motion.

Let us now consider his three laws of planetary motions. He had always felt that there was some profound law which controlled the motions of the planets round the sun. He sought to determine the number, the size, and the motion of the orbits of the planets. His materials were the invaluable records of Tycho Brahe's observations, and his own knowledge of geometry. Tycho's observations on Mars, so it seems, passed into the hands of his pupil and successor, Kepler, ~~who~~, unlike his master, was a Copernican. For two reasons Mars is the most difficult of the planets for which to construct tables in harmony with the observations. As Mars sometimes comes near the Earth, its movements can be determined with great precision and any irregularities become apparent. Besides this, the orbit of Mars diverges more from the circular form than that of any of the other planets. A very complicated geometrical system of epi-

cycles and eccentrics is required in order to represent the motion in satisfactory accordance with the facts.

Like Copernicus, Kepler followed in particular the movements of the planet Mars, these being sufficiently rapid to provide adequate data for testing purposes. His was essentially the method of trial and error. Every conceivable relationship between distance, the rate of motion, and the path of the planets he tested in the light of Brahe's results, only to reject them one after the other. Kepler tried hypothesis after hypothesis, but could not, by systems of epicycles, represent Tycho's observations with sufficient accuracy.

What was the correct orbit of Mars? He soon convinced himself that if it were a circle, then at any rate the sun could not be at its centre. After much labor he got a step farther. He noticed that when the planets' distance from the sun diminished, the planet went faster, and when the distance increased it went slower, and this brought him to the idea that the planet must sweep out equal areas in equal intervals of time. Suppose, therefore, that he were to represent the orbit by a circle, with the sun not at the center, would the planet under such conditions sweep out equal areas in equal times? He tested it for innumerable positions of the sun, but it never quite fitted.

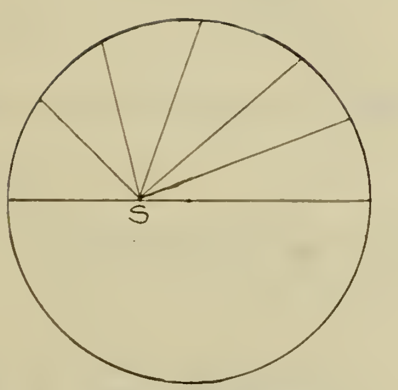


Fig. 5. Kepler's Test of a circular orbit with the sun not at the center.

Since his circular orbit (1) failed, he then tried other forms of curves, a daring innovation, as it was universally believed that the celestial motions must be composed of circular movements. Each different hypothesis involved an immense amount of labor, as all the calculations had to be started anew, and in Kepler's time there were no logarithm tables or other modes of simplifying numerical work.

At last, however, he hit upon the right solution. *Why not try an ellipse?* He tried it, with the sun in one of the foci, and it fitted beautifully with Brahe's observations. At last the long-sought secret was his. *The path of the planet is that of an ellipse with the sun at one focus, and the variations in speed are such that in equal times the planet sweeps out equal areas.* A reference to the figure below shows an

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(1) Hart, I. B. - Makers of Science - Oxford Univ. Press - Page 90.

ellipse with its two foci at S and T; the ellipse has (1) this property: if P is any point on the curve, the sum of the two lengths SP and TP is constant and equal to twice the length CA, C being the mid-point of ST. If the sun is supposed to be at S, the planet moves along the curve, being nearest the sun at A (perihelion) and furthest at B (aphelion). The length CA is called the semi-major axis of the orbit. A reference to Figure 7 below will at once show his idea regarding the speed of the planet. S, representing the sun, is at the focus of an ellipse A B C D.... This ellipse represents the orbit of any planet. Then obviously the distance of the planet will be a minimum at A and a maximum at F, and obser-

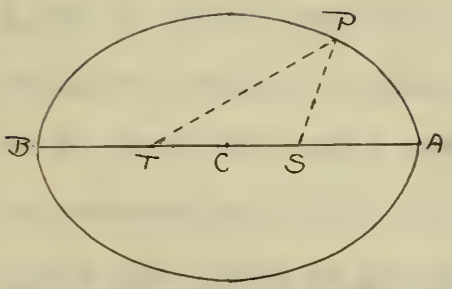

Fig. 6. The Ellipse.

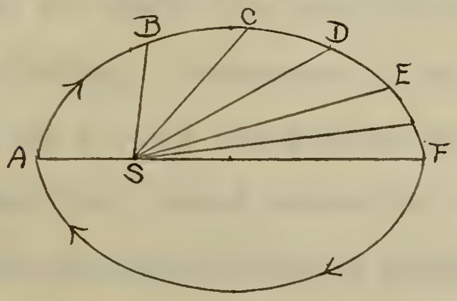

Fig. 7. Illustrating Kepler's First Two Laws of Planetary Motion.

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(1) Smart, W. M.- The Sun, The Stars and the Universe
Longman's Green Co. - page 31.

vation shows that the speed of motion is, on the contrary, a maximum at A and a minimum at F, so that if A, B, C, D, E, etc. be the positions of the planet at equal intervals of time, and if we join these points to S, then the areas swept through by the planet will be respectively ABS, BCS, CDS, DES, and EFS, and these areas are found to be exactly equal. (1)

The apparent simplicity of Kepler's discovery is apt to conceal the enormous mathematical difficulties with which he had to contend. Firstly, a single observation of a planet merely gives its direction at a particular instant with reference to the general background of the stars. Secondly, the observations are made from the earth, which is itself moving along an orbit round the sun. Thirdly, Kepler lacked the ordinary logarithmic aids to calculation with which every schoolboy is familiar. It would be a very simple matter to prove the areas in Figure 7 equal by means of calculus. Otherwise it is almost as complicated for us as it was for Kepler to prove that the planets sweep out equal areas in equal intervals of time. Some time we might like to convince ourselves of this profound truth. For the present, we might be satisfied with the following simple device. Thus in Figure 8 be-

Hart, Ivor B. - Makers of Science - Oxford Univ. Press
Pages 90 - 91.

low a planet will describe the large ∡ A' SP' in the same time as the small ∡ ASP, because the area A' SP' equals the area ASP. A convenient, although crude, device is to draw such a representation of the planetary motions on cardboard and cut out the shaded areas. They should have the same weight. This might serve as a simple proof that the areas are equal and thus convince us of the truth of Kepler's second law.

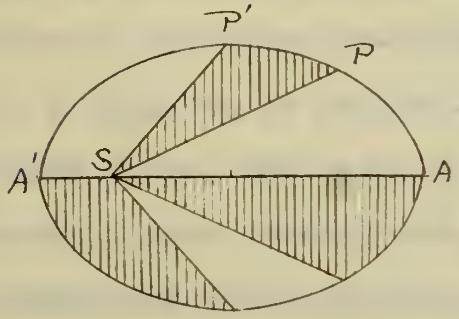

Figure 8.

Kepler was fully entitled to his triumph, but his work was not yet complete. He had yet to unravel the relationship which he felt existed between the distances of the different planets and their average speeds around the sun. Why he felt such a relationship to exist we can hardly say, but he had a feeling that simple mathematical laws were traceable in all natural phenomena. He had little to go upon. It was part of his genius that he felt intuitively that not only was there such a relationship, but also that he would find it out sooner or later. And find it out he did, though later rather than sooner. Let him speak for himself: "What I prophesied two and twenty years ago,

when I discovered the five solids among the heavenly orbits--what I firmly believed long before I had seen Ptolemy's "Harmonies"--what I had promised my friends in the title of this book, which I named before I was sure of my discovery--what, sixteen years ago, I urged as a thing to be sought--that for which I joined with Tycho Brahe, for which I settled in Prague, and for which I have devoted the best part of my life to astronomical contemplations, this at length I have brought to light. It is not eighteen months since the first glimpse of light reached me, three months since the dawn, very few days since the unveiled sun, most admirable to gaze upon, burst out upon me.---

If you would know the precise moment, the idea first came across me on the 8th of March of this year 1618, but, chancing to make a mistake in the calculations, I then rejected it as false. I returned to it again with new force on May 15; and it has dissipated the darkness of my mind, by such an agreement between this idea and my seventeen years' labor on Brahe's Observations, that at first I thought I must be dreaming, and had taken my result for granted in my first assumptions. But the fact is certain, that *the proportion existing between the periodic times of any two planets is exactly the sesquiplicate proportion of the mean distances of the orbits*." (1)

(1) Hart, I. B. Makers of Science - Oxford Univ. Press - page 91.

And now let us put this into simple language. Kepler's discovery was this: *that for all planets, the squares of the times of revolution round the sun are as the cubes of the mean distances.* Let us try it, for example, in the case of Mars.

The data are as follows, taking the Earth's distance as one unit, and the Earth's time (i. e. time of one complete revolution) also as unity:

Earth's Distance from Sun .... 1.0000 units
Mars' Distance from Sun .... 1.5237 units
Earth's Period of Revolution........... 1.0000 years.

Required the period of revolution for Mars. We have, by Kepler's Third Law

$$\frac{(\text{Mars' Period})^2}{(\text{Mars' Distance})^3} = \frac{(\text{Earth's Period})^2}{(\text{Earth's Distance})^3}$$

$$\text{Hence } \frac{(\text{Mars' Period})^2}{(1.5237)^3} = \frac{1^2}{1^3}$$

$$\text{Mars' Period of Revolution} = \sqrt{1.5237^3}$$

$$= 1.08808 \text{ years}$$

which is true.

And so with all planets. The meaning of this third law will be seen from the following table, in which "a" is the mean distance from the Sun, the Earth's mean distance being taken as 1, and T is the time in years of its revolution.

	Mercury	Venus	Earth	Mars	Jupiter	Saturn (1)
a	.387	.723	1	1.524	5.203	9.539
a^3	.058	.378	1	3.54	140.8	868.0
T	.241	.615	1	1.881	11.86	29.46
T^2	.058	.378	1	3.54	140.7	867.9

The agreement of the figures in the second line with the corresponding figures in the fourth line constitutes the third law.

To Kepler this was simply an interesting numerical relation, apparently independent of either of the first two laws. We shall see later that these empirical laws are the consequence of the great universal law of gravitation, the offspring of Newton's genius.

This then was Kepler's life work, and no more fitting tribute to his memory could be given than to impress upon you a restatement of his laws:

Law 1. All the planets move round the sun with elliptic orbits with the sun at one focus.

Law 2. The radius vector, or line joining the planet to the sun, sweeps out equal areas in equal intervals of time.

Law 3. For all planets, the square of the time of one complete revolution (or year) is proportional to the cube of the mean distance from the sun.

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(1) Dyson, F. W. - Astronomy - E. P. Dutton Co. Page 42.

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APPRECIATION UNIT II

THE MATHEMATICS USED BY NEWTON IN DETERMINING THE UNIVERSAL LAW OF GRAVITATION.

THE LAW OF UNIVERSAL GRAVITATION

"Towering head and shoulders above all his contemporaries, a veritable giant among the giants, a man whose intellect and whose contributions to knowledge are incomparably greater than those of any other scientist of the past, was that prince of philosophers, Sir Isaac Newton." (1)

Soon after his graduation from Trinity College, Cambridge, Newton turned his thoughts to those speculations on the subject of gravity which resulted in his famous enunciation of the inverse square law of universal gravitation, and enabled him to give a definite proof of the three laws of planetary motion discovered, but left unexplained, by Kepler. The problem of gravity was not a new one. Speculations on the subject had interested almost every philosopher of note from the time of Plato. What had been called the power of gravity was familiar to all. It remained to discover the law which this power obeyed. Newton, like his predecessors, found the problem of fascinating interest.

It will be remembered that Kepler, after a lifework of patient analysis of Tycho Brahe's famous record of observations, had enunciated his three famous

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(1) Hart, Ivor B. - Makers of Science - Oxford Univ. Press - page 140.

laws of planetary motion. The third of these laws, it will be recalled, was that for every planet, the cube of the distance is proportional to the square of the periodic time; or, expressing this mathematically, $\frac{r^3}{T^2}$ is constant for all planets. Why should this be so? This was what Newton asked himself.

To understand this further, let us vary Kepler's statements a little. He told us that the orbits of the planets are elliptical. It will simplify the problem without upsetting the argument in any way, if we adhere to circles. Suppose the orbits are circular. What 'binding' force is there in a body like the sun which will keep a planet moving constantly around it in a circular orbit? One finds a similar problem in the twirling of a mass in a circular path by means of a string. Referring to the figure below, suppose the string is held at C, then initially, taking the point A as the mass in question, its direction of motion is along the tangent AB. That is to say, unless otherwise restrained, it will 'fly off at a tangent,' as the popular phrase goes. But it __is__ so restrained--by the string AC, which, by

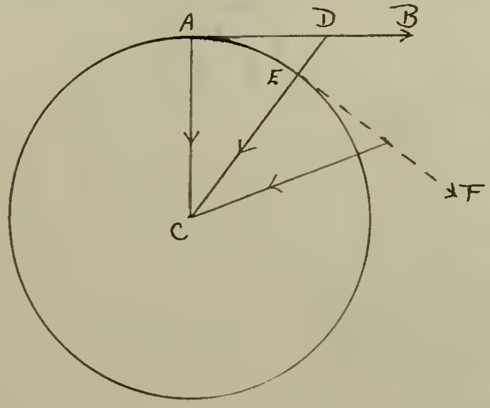

Figure 9.

tending to pull the mass into itself, brings it in a given time to E on the circle instead of to D on the straight line. In effect, in the given time in question, it has pulled the mass inwards towards the center through a distance DE. The new direction of motion is now EF, the tangent at E, and the whole argument continually repeats itself, and the force along the string directed towards the center C constantly pulls the body out of its 'straight line' tendency and keeps it moving in a circular path or orbit.

Let us now apply a little mathematics and elementary mechanics to the problem, and it will then be seen why Newton was seeking for the existence of a constant force directed towards the center.

Perhaps you would like to start, as Newton did, from the simple terrestrial phenomenon of a falling body. To explain what keeps the moon revolving round the earth let us consider a simple case of a falling body. It had long been realized that when a stone is allowed to fall some force is responsible for the stone's motion in the direction of the center of the earth. This force evidently operated at the tops of the highest towers

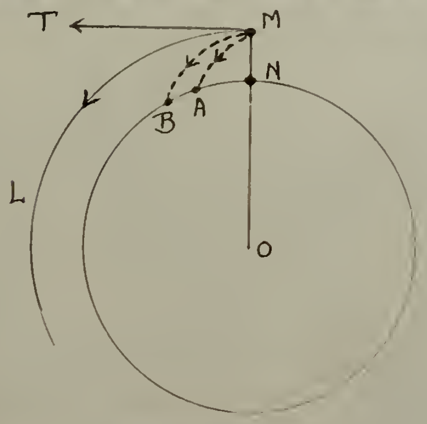

Figure 10

and on the summits of the highest mountains. Did it operate as far away as the moon? This question Newton succeeded in answering in the affirmative: the agency that was responsible for the fall of a stone or of the traditional apple was the same agency that kept the moon revolving around the earth. Suppose a gun is (1) fired horizontally in the direction MT from the top, M, of an eminence at a height NM above a level plain (Figure10): the bullet will describe a trajectory such as MA. If the velocity of the projection is increased, the point of impact on the earth's surface will be at B, further from N than A. If the velocity be still further increased the trajectory will be lengthened out still more. These are facts which we can regard as obvious deductions from experience. We are led to infer that when the velocity reaches a particular value (about 5 miles per sec.) the bullet will not strike the earth at all, but will revolve round it in a circle of radius OM, becoming in fact a satellite like the moon. Now when the bullet is describing a trajectory MA its motion is partly horizontal and partly vertically downwards, the latter part being due to its fall from an altitude MN. So when the bullet is describing the circle of radius OM it is continually

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(1) Smart, W. M. - The Sun, the Stars, and the Universe - Longman's, Green & Co. - pages 34 - 35.

falling towards the centre of the earth, as well as moving forward horizontally at each point of the orbit. Consider the situation one second after projection from M with the horizontal velocity of 5 miles per second (Figure 11). The horizontal velocity would take the bullet to Q, MQ being 5 miles, in one second. But it is also falling, so that at the end of one second it will have fallen by a distance QR, R being on the line joining Q to O. Now, if NF is the horizontal line through N meeting OQ in F, the distance NF is practically 5 miles, for NM is very small compared with the radius ON (the altitude NM has been vastly exaggerated in the figure). From the known radius ON or OG of the earth the distance FG can be easily calculated: it is found to be 16 feet. Now, as we have supposed the bullet to be describing a circular orbit of radius OM or OR, it follows that QR is 16 feet. But this is the distance through which a stone will fall from rest in one second at the surface of the earth. The consequence is that, although the bullet is continually

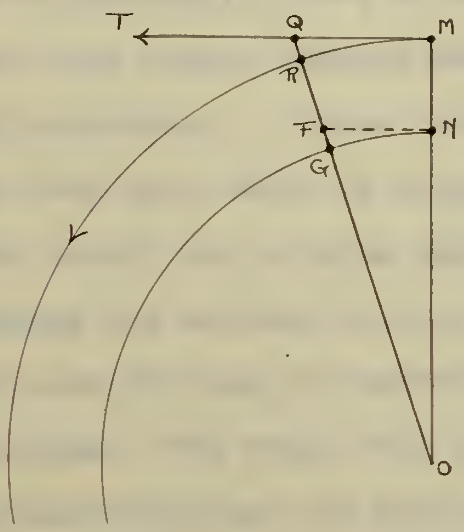

Figure 11.

falling, it remains at the same distance above the earth's surface, owing to the earth's curvature and the particular velocity of projection which we have assumed. Thus from the known phenomena of falling bodies, a circular orbit of a body moving close to the earth's surface can be adequately explained. Now what makes a stone fall? To answer this question we must invoke the aid of Newton's laws of motion.

In his famous book, the Principia, Newton enunciated three laws of motion. The first of these tells us that in a world of no forces, if such a thing could be imagined, a body would do only one of two things; it would either be at rest, or, if moving at all, it would move continually in a straight line with uniform speed; and this simply because there are no forces to make it do otherwise. A force continuously applied to a body at rest will start it moving with continually increasing speed, and a force applied to a moving body will change its motion; that is to say, it will speed it up, or slow it down, or swerve it from its original direction. The force will in fact produce in the body an acceleration (or, of course, a retardation which mathematically is merely a negative acceleration). This was Newton's first law. His second was in part nothing but a common-sense law, and said in effect that the amount of acceleration produced in a body depends upon the magnitude of the force; a big force must have a big ef-

fect--a small force a small effect. The acceleration (or rate of change of motion) produced, then, is proportional to the impressed force. This is algebraically the $F = m.a$ formula well known to all students of elementary mechanics.

Now, a falling body does not move with uniform velocity: it is found by experiment that a stone, dropped from rest, falls 16 feet in the first second and 48 feet in the next second. Therefore a force is acting on it which we may describe as a pull. This is a quite definite result so far as motion at the surface of the earth is concerned, and we can explain the circular orbit of the bullet above by virtue of this force continually acting on it in a direction always pointing towards the centre of the earth. Now let us consider the moon's orbit around the earth. The same principles apply, but what is the relation of the force which keeps the moon in its orbit and prevents its flying off at a tangent, to the force at the surface of the earth which makes a stone fall?

Let us return to the problem of a body being whirled round a circular orbit (Fig. 9) about a center C. We have seen that the circular orbit is produced through the exertion of a constant force along the string directed towards the center. As a matter of common-sense, it must be obvious that the swifter the speed (let us call it v) with which the body tends to

'fly off along the tangent', the greater must be the controlling force to give it the necessary change of speed of motion (or acceleration) inward to keep it in its circular path. Both Huyghens and Newton showed in fact that the acceleration required must be $\frac{v^2}{r}$ when r is the radius of the circle, and v the velocity of the mass. The investigation of this problem would be out of place here--let us accept the result and use it as required. Now since the force $F = m.a$, we have by substitution

$$F = \frac{m\ v^2}{r}$$

This gives us the measure of the centripetal force, as it is called, necessary to hold any given mass "m" moving with a speed "v" in a circular orbit of radius "r", and this force is always directed towards the center of the orbit.

We are now in a position to see a little light with regard to Kepler's third law of planetary motion. He talks of the periodic time; he means, of course, the time for one complete revolution. If the speed is "v", and the circumference $2\pi r$, then obviously we must have the periodic time T, for

Since, distance gone = time taken x speed

Then, time taken = $\frac{\text{distance gone}}{\text{speed}}$

Or, speed = $\frac{\text{distance gone}}{\text{time taken.}}$

Whence, $V = \frac{2\pi r}{T}$, by substitution.

Now, since $F = \frac{mv^2}{r}$, we can substitute $\frac{2\pi r}{T}$ for v.

Hence $F = \frac{m\left(\frac{2\pi r}{T}\right)^2}{r}$ (substitution)

$F = \frac{m.4\pi^2 r^2}{T^2} \cdot \frac{1}{r}$ (squaring and inverting divisor)

And $F = \frac{m\, 4\pi^2 r}{T^2}$ (simplifying)

This result is an expression for the constant force required to hold a body in its circular orbit.

Now compare this with the constant expression required by Kepler's third law --namely, $\frac{r^3}{T^2}$. To get a correspondence between these two expressions, we want an r^2 term to come into the numerator of the former one. We must remember that the expression

$$F = \frac{m\, 4\pi^2 r}{T^2}$$

is concerned with the problem of a body whirled round a circular path at the end of the string, the force F being the inward force exerted along the string. Kepler's expression, on the other hand, is for the actual

planet, where there are no strings connecting the body to the central sun.

We want, in this latter case, a substitute for the force along the string. Let us assume, therefore, that, in the course of the planet moving round the sun in a circular orbit (we are still regarding the orbits as circular instead of elliptical), there is a constant force directed towards the sun, exerted, so to speak, along an imaginary string. If we can take this force as being inversely proportional to the square of the radius, we shall have achieved our object of a proper correspondence between these two expressions. Let us see how it works out.

Let us suppose that there is a law according to which a body is naturally attracted to a 'central' mass with a force which is inversely proportional to the square of the distance between them, and is directly proportional to the masses of the two bodies. To express this law algebraically:

Let M = the mass of the central or attracting body.
m = the mass of the body itself.
v = the speed
r = the radius of the circular orbit.
F = the force
K = a constant.

Then we have:

(1) $$F = K \cdot \frac{M\,m}{r^2}$$

But we have seen that

(2) $$F = \frac{m\,4\pi^2\,r}{T^2}$$

Then, since quantities equal to the same quantities are equal to each other, we see that equation (1) equals equation (2).

Hence $$\frac{m\,4\pi^2\,r}{T^2} = \frac{K.M.m}{r^2}$$

$$m\,4\pi^2\,r^3 = K.M.m.T^2$$ (clearing of fractions)

Then $$4\pi^2\,r^3 = K.M.T^2$$ (dividing by m)

Or $$(4\pi^2)\,(r^3) = (M.K.)\,(T^2)$$

Hence $$\frac{r^3}{T^2} = \frac{M.K}{4\pi^2}$$ (dividing by $4\pi^2\,T^2$)

(This last statement may also be explained by the statement "If the product of two quantities equal the product of two others, one may be made the extremes in a proportion and the other the means).

If we examine this last statement, namely,

$$\frac{r^3}{T^2} = \frac{M\,K}{4\pi^2}\ ,$$

we see that the right-hand side of this equation is a constant quantity, since K is the constant factor of proportionality already mentioned, and M is the mass of the central attracting body.

$$\text{Therefore} \quad \frac{r^3}{T^2} = \frac{(\text{Distance})^3}{(\text{Periodic Time})^2} = \text{Constant.}$$

This is nothing more nor less than Kepler's Third Law.

"It follows then, from a consideration of the elementary mechanics of central orbits that it would be quite easy to explain Kepler's third law of planetary motion, if there could be found to exist a force of attraction towards the central body whose magnitude is inversely proportional to the square of the distance." (1)

For the sake of simplicity we have confined ourselves to the case of circular orbits only. But, although the mathematics is more complicated for elliptical orbits, the result is the same, so the problem is perfectly general. Although there seems to be no foundation for the well-known story of the falling apple, it is extremely probable that Newton must at least have considered as significant the every-day experience of the behavior of an object such as a freely-falling stone.

Why was such an experience significant? It was because it enabled Newton to link up certain facts.

(1) Hart, Ivor B. - Makers of Science - Oxford Univ. Press. page 158.

Here was an example of a force exerted by the earth on the stone. So far as the planets are concerned, the force he sought was to be exerted by the sun.

Was it possible, thought Newton, that in every case the force of attraction was inversely proportional to the square of the distance? In that case would it account too for the motion of our own moon around the earth? For, surely, if the earth attracts the stone, it must also attract the moon.

Here was a splendid chance for applying a test. The earth's pull or attraction falls off as the square of the distance. The moon was known to be distant (1) sixty times the earth's radius from the earth's centre. Hence the attraction exerted by the earth on the moon must be $\left(\frac{1}{60 \times 60}\right)$ of the attraction it will exert on a body at its own surface (since a body on the earth's surface is $\frac{1}{60}$ of the distance of the moon from the earth's centre).

We have already discovered that a body on the (2) earth's surface is pulled to it (i.e. drops) a distance of 16 feet (or more exactly 193 inches) in one second. Therefore the moon should 'drop' a distance of $\left(\frac{16}{60 \times 60}\right)$ feet in one second.

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(1) Brodetsky, S. - Sir Isaac Newton - Methuen & Co. page 50.

(2) Smart, W. M. - The Sun, the Stars, and The Universe - Longman's Green & Co. - page 36.

We can test this for we know by observations exactly how long the moon takes to go once round, and we know the distance between the earth and moon. Therefore we could draw a diagram (like Figure 9 above) to represent the moon's motion round the earth C, and we could calculate how much arc AE (in Figure 9) could be traversed by the moon in one second. All we have to do is to calculate the 'drop' DE, and assure ourselves that it works out to $\left(\frac{16}{60 \times 60}\right)$ feet.

By a tragedy of misfortune, for which Newton was not to blame, his test was doomed at first to disappointment. The accuracy of the result depended upon the accuracy of the value taken as the radius of the earth. Newton could only take the accepted value of the times, and it was, alas, wrong. It was assumed to be 3,436 miles, whereas the correct value is 3,963 miles. Consequently, instead of getting the required drop for the moon of $\left(\frac{16}{60 \times 60}\right)$ feet in one second, he only got $\left(\frac{14}{60 \times 60}\right)$ feet in one second.

Newton's disappointment was keen. He had found the acceleration of the moon to be 0.00775 feet per (1) second per second, while Kepler had found it to be 0.00895 feet per second per second! The latter exceeds the former by nearly 16 per cent! Since he had no reason to suspect his data, Newton found himself

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(1) Brodetsky, S. - Sir Isaac Newton - Methuen & Co. page 50.

obliged to abandon the basic assumption of an inverse square law.

About 16 years later, at a meeting of the Royal Society, Picard communicated the results of his more accurate determinations of the value of the earth's radius. Newton was present, and made a careful note of Picard's results. He recalled his work and hopes of 16 years before. "He hurried home and routed out (1) his old papers and calculations on the subject. His excitement at the anticipation of complete success to his theory was too much for him. He was too agitated to finish his calculations, and he got a friend to do them for him. The result was a complete triumph. The value for the moon's 'fall' per second was exactly $\left(\frac{16}{60 \times 60}\right)$ feet. At last he had discovered both the true law of universal gravitation and the true explanations of Kepler's laws of planetary motion. And all this would have been definitely achieved sixteen years before but for an inaccurate determination for which he was not responsible."

In spite of the magnificence of this achievement, Newton said nothing about his discovery. He was always chary of publicity. How then did his discovery ultimately receive the wide publicity which importance demanded? The story is an interesting one. It was

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(1) Hart, I. B. - Makers of Science - Oxford Univ. Press - page 159

Edmund Halley to whom the world is indebted for bringing to light what must always remain one of the most important documents in the world's scientific history. When Newton had told Halley that he had calculated the curve described by a planet, he promised to send him his calculations. Halley at once communicated them to the Royal Society. As Halley proudly put it, he was the "Ulysses who produced this Achilles."

The importance of Newton's work was by no means lost on the Royal Society. They wrote to Newton and asked permission to publish his researches. Newton consented and as a consequence the world has his epoch-making Principia, or Mathematical Principles of Natural Philosophy. It is without exception one of the most important works in natural philosophy extant.

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APPRECIATION UNIT III

THE MATHEMATICS USED IN THE DISCOVERY OF NEPTUNE

MATHEMATICS USED IN THE DISCOVERY OF NEPTUNE.

The existence of Neptune in the sky was revealed by the sole power of numbers. It is one of the most eloquent witnesses of the power of mathematics. This planet distant more than 2,700 millions of miles from the earth, is absolutely invisible to the naked eye. The perturbations manifested by the planet Uranus permitted the mathematician to say that the cause of these irregularities was an unknown planet which revolved beyond Uranus and which, to produce the effect observed, should be found at a certain point of the starry sky. A telescope was directed toward the point indicated, the unknown was searched for, and in less than an hour it was found!

Suppose we investigate and find out just how this magnificent piece of work was accomplished. Let us discover, first of all, just how this problem arose. Maybe a few fundamental facts from astronomy will help us.

If the planets only obeyed the action of the sun they would describe elliptic orbits around it. But they act on each other; they likewise act on the central star, and from these various attractions irregularities result. These irregularities are called perturbations. Astronomers construct, in advance, tables of the positions of the stars in the sky, in order to know where they should be found exactly, or to verify their motions, or for the numerous applications of astronomy to geography and navigation.

More than a century ago, an able French astronomer, M. Alexis Bouvard, applied himself to the task of making a refined investigation of the motion of Uranus. Calculating, in 1820, the positions of Jupiter, Saturn, and Uranus, he ascertained that the theoretical positions given by his tables agreed perfectly with the most recent observations at that time for the first two planets, while for Uranus there were inexplicable differences. From 1820 to 1840 these discrepancies worried all astronomers for the differences between the calculated and observed positions of Uranus went on increasing: it was 20" in 1830, 90" in 1840, 120" in 1844, and 128" in 1846. To a man of the world, an artist, or a merchant, this difference would have seemed so slight that he would not even have noticed it. In fact, if there were two adjacent stars in the sky at this distance from each other, excellent sight would be required to separate them clearly. But to an astronomer such a difference becomes altogether intolerable.

In 1843, John C. Adams, while a student at St. John's College, Cambridge, worked in retirement on the hypothesis of an exterior planet as the cause of the disturbances of Uranus. In October, 1845, he forwarded to Airy, Astronomer Royal, some provisional elements for a planet revolving around the sun at such a distance and of such a mass as he thought would account for the observed perturbations of Uranus. This was virtually

the solution of the problem. It so happened that in the summer of 1845, Le Verrier, of Paris, turned his attention to the anomalous movements of Uranus and located, on paper, by mathematical calculation, the position of this new planet in the sky. On August 31, 1846, Le Verrier announced to the Academy of Sciences that the planet should be found at longitude 326°. In September, Dr. Galle, of the University of Berlin, searched for the planet. He directed his telescope toward the point indicated, and perceived a star which was not on his chart, and which showed a perceptible planetary disc. Its position in the sky was 327° 24'; calculation had indicated 326° 32'. The longitude had therefore been precisely stated to within one degree!

We may very easily understand the disturbing action of an exterior planet on the position of Uranus by an examination of this figure which shows the positions of the two planets from the discovery of Uranus to that of Neptune.

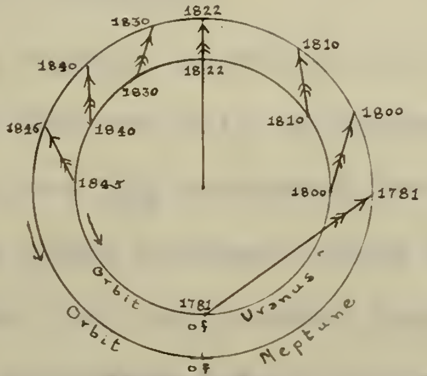

Figure 12.

Disturbances of Uranus by Neptune. (1)

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(1) Flammarion, C. - Popular Astronomy - D. Appleton & Co. page 464.

This diagram shows the paths of Uranus and Neptune from 1781 to 1840 and will help to illustrate the direction of the perturbing action of the latter planet on the former.

We see that from 1781 to 1822 the influence of Neptune tended to draw Uranus in advance of its place as computed independently of exterior perturbation.

In 1822 the two planets were in heliocentric conjunction, and the only effect of Neptune's influence was to draw Uranus farther from the Sun without altering its longitude.

From 1822 to 1830 the effect of Neptune was to destroy the excess of longitude accumulated from 1781, and after 1830 the error in longitude changed its sign and for some years following Uranus was retarded by Neptune. By 1840 it had fallen 128" behind its place as predicted by Bouvard's tables.

Professor Adams gave the following explanatory comment on the above diagram:

"The arrows rightly represent the direction of the force with which Neptune acts on Uranus taken singly but the diagram does not represent the direction of the disturbing force which Neptune exerts on Uranus relatively to the Sun, and this latter force we must take into account in computing the planetary perturbations. To find this disturbing force, we must take the force (1)

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(1) Chambers, G. F. - Handbook of Astronomy - Oxford
Clarendon Press - page 257.

of Neptune on the Sun, reverse its direction and then compound this with the direct force of Neptune on Uranus.

"Thus, in the figure below, if S denotes the Sun, U represents Uranus, and N stands for Neptune, the force of Neptune on Uranus will be in the direction UN and will be proportional to $\frac{1}{(UN)^2}$, and the force of Neptune on the Sun will be in the direction SN and will be proportional to $\frac{1}{(SN)^2}$. Hence if we produce NS, if necessary to V, and take $NV = \frac{(UN)^3}{(SN)^2}$, then the reversed force of Neptune on the sun will be represented by NV, provided the direct force of Neptune on Uranus be represented by UN. Hence the disturbing force of Neptune on Uranus relatively to the sun will be represented on the same scale in magnitude and direction by UV, the direction being indicated by the arrow in the figure and the magnitude of the disturbing force being proportional to $\frac{UV}{(UN)^3}$.

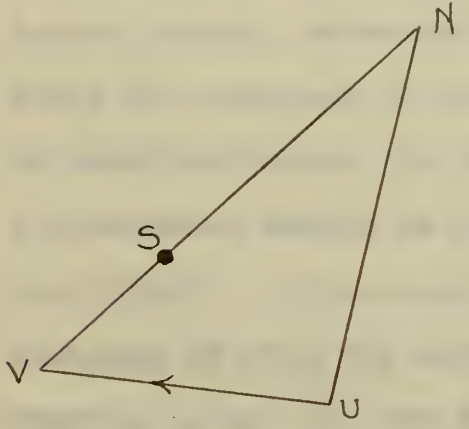

Figure 13.

"It is not possible to state the effect of Neptune's action on the motion of Uranus in such simple terms as you have attempted to do, since it is necessary to take into account the action of Neptune in order to find the correct elements of the orbit of Uranus and consequently the corrections of the assumed elements must be taken as additional unknown quantities which must be determined simultaneously with perturbations depending on Neptune."

You can readily see, that a tremendous amount of calculation was performed by both Adams and LeVerrier. LeVerrier even went so far as to reconstruct and verify the tables of Bouvard, but the errors found did not account for the differences found. It is also remarkable to note that, in the presence of this well established inconsistency, LeVerrier did not for a single moment doubt the accuracy of the law of universal gravitation. He boldly advanced the hypothesis of a planet acting in a continuous manner on Uranus and changing his motion very slowly. He supposed that this would be at the distance 38 from the series of Titius, and consequently revolve around the sun in 217 years; and on this hypothesis he calculated what position it should have in the sky behind Uranus in order to produce by its attraction the observed differences, and what the mass should be to explain the amount of deviation. He then recalculated the orbit of Uranus, taking into account

the perturbations thus produced by the disturbing planet and found that all the positions agreed with the theory. From that time the problem was practically solved. Considered from the point of view of practical astronomy, it was but a simple exercise in calculation and the most eminent astronomers saw in it nothing else! It was only after its verification, its public demonstration--it was only after the visual discovery of Neptune--that grandeur and magnificence of the discovery was realized.

The intellectual grandeur of this discovery will be best appreciated by placing in juxtaposition the observed longitude of the new planet when telescopically discovered, and the computed longitudes of Adams and Le Verrier.

HELIOCENTRIC POSITIONS (1)

Observed	326° 52'
Computed by Adams	329° 19'
Computed by LeVerrier	326° 0'

Adams $C - O = 2^\circ\ 27'$

Le Verrier $C - O = -0^\circ\ 52'$

From this it will be seen that LeVerrier's computation proved to be slightly more accurate, a fact which in no respect militates against the equality of the merits of the two great mathematicians.

It should be stated, however that some of the re-

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(1) Chambers, G. F. - Handbook of Astronomy - Oxford
Clarendon Press - page 258.

sulting data derived by LeVerrier will not stand comparison with the elements as we know them to be today. This is due to the fact that he used 36.154 to be the distance from the sun, while in reality the distance is 30.055. This makes the period of revolution of Neptune 164 years and 281 days. Such is the year of the Neptunians!

It is interesting to note that, for some time past, there seems to have been unmistakable evidence that Neptune is deviating from its predicted orbit. The method of Adams and LeVerrier have been applied again, and it is expected that telescopic detections will soon follow as surely as in the case of Neptune. Although it may seem that we are reaching out into the world of speculation and surmise, it is at present believed probable, in consideration of mathematical calculations concerning the orbit of Neptune, that some other planet, not Neptune, marks the outermost limit of the planetary system.

"Such", in the words of Hind, "is a brief history (1) of this most brilliant discovery, the greatest of which astronomy can boast, and one that is destined to perpetual record in the annals of science--an astonishing proof of the power of the human intellect".

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APPRECIATION UNIT IV

MATHEMATICS AND NAVIGATION

1. Pilotage
2. Dead-reckoning
3. Nautical astronomy

NAVIGATION

Navigation is not a branch of mathematics but it cannot exist without mathematics. It is one of the most practical applications of trigonometry--both plane and spherical.

Navigation is an art rather than a science. It is the art of finding the geographical location of a vessel at sea and of guiding it to a desired port or position. Let us consider the three parts--pilotage, dead-reckoning, and nautical astronomy.

A. Pilotage

This deals with the art of guiding the vessel when it is within sight, depending on landmarks; for example, the determination of the distance from a lighthouse, near the entrance of the harbor, with a danger zone of rocks about the port. The lighthouse may be identified by its markings and the Coast Pilot gives the height of the lighthouse above high and low tide.

Suppose an officer wants to know if he is keeping far enough away from the danger zone of rocks. He assumes an observation at the surface of the water and gets an angular distance to the top of the lighthouse.

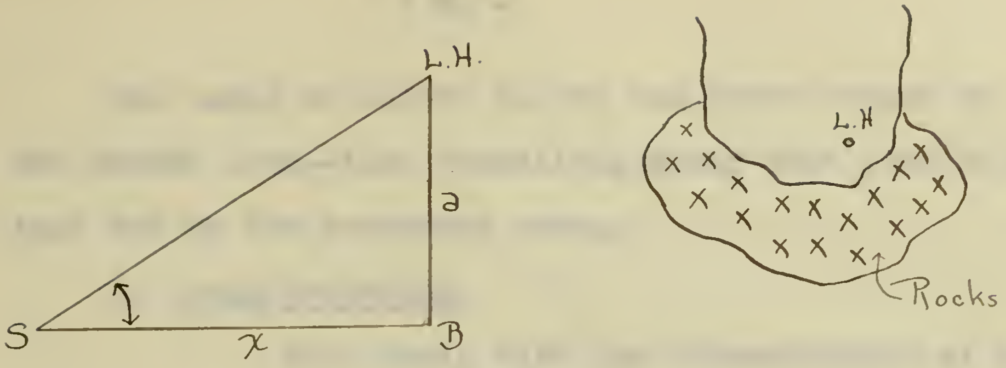


Figure 14.

In this figure, "a" is known, ∡ S is also known, while "x" is unknown.

TanS gives us x. Since

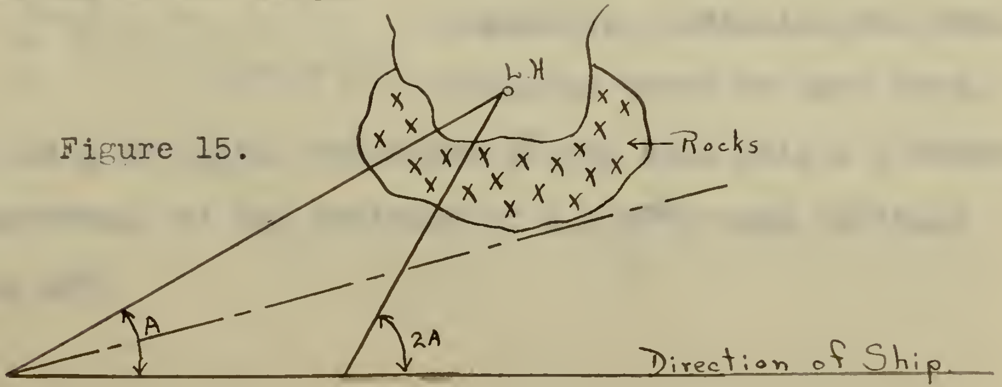
$$\text{Tan } S = \frac{a}{x}$$

Then

$$x = \frac{a}{\text{Tan } S}$$

The size of the danger zone is given on his chart and therefore as the boat comes up he measures the angle between the direction of the ship and the lighthouse horizontal angle and continues measuring until he has ∡ 2 A, i.e., doubling the angle on the bow. Then what is the distance from the lighthouse? The distance he has gone is the distance required. The angles are measured from the sextant and the distance is given on the log of the ship.

Figure 15.



The angle of safety is the one above formed by the dotted line--i.e. travelling along that line he can just get by the dangerous rocks.

B. Dead Reckoning.

This deals with the determination of position at sea and how to reach the desired port. The compass and the log of the boat are used in dead reckoning. Direction is determined by the compass and the boat can be turned in the required direction. Distances are measured by the log which shows the rate and distance.

1 mile = 1 minute of arc (distance)

1 knot = rate of 1 nautical mile per hour.

For small areas consider the surface flat, therefore the ocean is flat.

Suppose the ship is going from A to B, apparently north, in the figure below. Required to determine the "distance made good"--that is, the longitude and latitude of B, if we know the longitude and latitude of A. We can solve and get AH, which, therefore, indicates the total distance north we have gone.

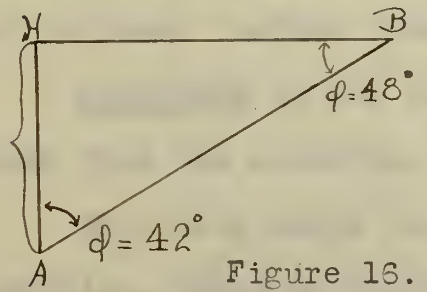


Figure 16.

If AH = 60 miles, therefore 1° for each mile = 1 minute. Therefore, if the latitude of A = 42°, then latitude B = 48°.

To find the longitude is harder. The meridians converge and therefore a nautical mile is not the distance on a small circle, but on the equator or meridian and it is the problem of the mariner to avoid this difficulty. Plane sailing is the method used for this purpose.

Plane sailing is the art of navigating a ship upon principles deduced from the supposition of the (1) earth's being an extended plane, on which the meridians are all parallel to each other.

The course is the angle which the line described by a ship makes with the meridian sometimes reckoned in degrees, sometimes in points.

Distance is the way or length a ship has gone on a direct course in a given time.

Difference of latitude is the distance a ship has made north or south of the place sailed from, or the portion of the meridian contained between the parallels of latitude 'sailed from' and 'come to'.

Departure is the east or west distance a ship has made from the meridian.

If a ship sails due north or south, she sails on a meridian, makes no departure, and her distance and difference of latitude are the same. If she sails east or west she goes on a parallel of latitude and makes no difference of latitude and her departure and distance

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(1) Bowditch, N. - New American Practical Navigator - E. & G. W. Blunt - page 52.

are the same.

The difference of latitude and departure make the legs of a right triangle, the hypotenuse of which is the distance the ship has sailed; the perpendicular is the difference of latitude counted on the meridian; the base is the departure, which is easting or westing counted from the meridian; the angle opposite the base is the course, or the angle a ship makes with the meridian; the angle opposite the perpendicular is the complement of the course, which being taken together make 8 points or 90°.

One type of plane sailing is parallel sailing--that is E ⟶ W. This is used a great deal because of the following reasons:

1. The navigation is very simple.
2. Water transportation is comparatively inexpensive and the extra time doesn't make much difference.
3. Easy sailing--the shortest methods of passage may take boats into ice fields. To overcome this, sail to parallel latitudes, sail along this parallel, and then go down to port.

It is erroneous to suppose the earth a plane because the earth is nearly a sphere, in which the meridians all meet at poles, therefore the distance of any two meridians measured on a parallel of latitude decreases in proceeding from the equator to the poles.

It is necessary to change nautical miles to difference of longitude. If on the equator, this is easy, for the longitude tells the number of miles we have gone.

For example, 32 miles = 32 minutes of arc, for each mile = 1 minute.

If we started from the longitude 62°, and travelled 32 miles, we are on the equator 62° 32'.

Let us see if we can change this to a parallel of known latitude. Look in the globe and we shall see a condition of this sort.

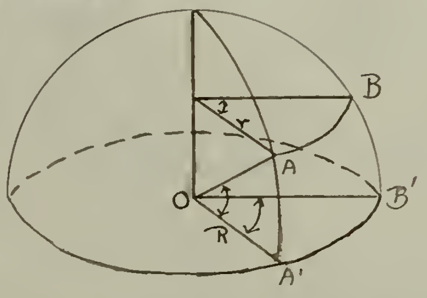


Figure 17.

The number of degrees in the arc of AB equals the number of degrees in the arc A'B', because the plane angles are equal.

Suppose R = radius of earth.

r = radius of small circle.

Then $$\frac{AB}{A'B'} = \frac{r}{R}$$

If we know AB in nautical miles, we can get the angle at the center and that is the difference of longitude change. If we know r, R, and AB, we can get A'B',

the difference of longitude.

We know the latitude on which we are sailing. Draw a line from A to the center O and the latitude is the angle formed by $\measuredangle AOA'$.

$\measuredangle AOA' = \phi$. Draw a perpendicular from A to A'O. (See figure below).

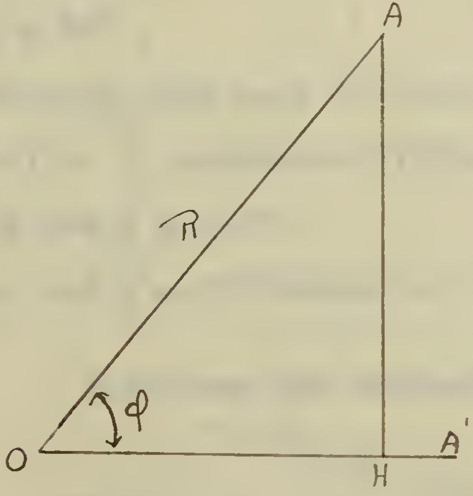


Figure 18.

$$\cos\phi = \frac{OH}{R}$$

$$OH = R\cos\phi$$

$OH = r$ (Parallels included by parallels)

$A'B'$ = difference of longitude of the two places.

ϕ = Latitude.

Latitude does not change.

$$\therefore \frac{AB}{A'B'} = \frac{R\cos\phi}{R}$$ (substitution)

$$\therefore \frac{AB}{A'B'} = \cos\phi$$

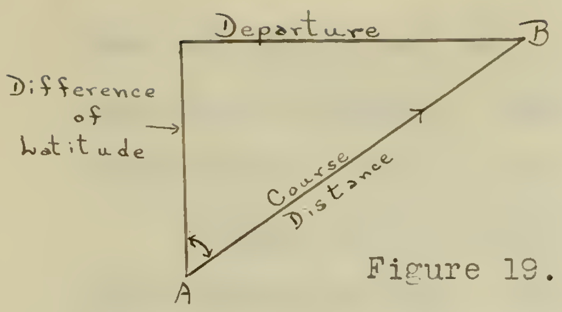
$$AB = A'B'\cos\phi$$

$$A'B' = \frac{AB}{\cos\phi}$$

$$\left\{\text{but } \frac{1}{\cos\phi} = \sec\phi\right\}$$

$$\therefore A'B' = AB\sec\phi.$$

To explain a little more in detail use the following triangle:


Departure
B
Difference of Latitude
Course
Distance
A

AH is along the meridian so transform miles to arcs. Not so for HB.

Figure 19.

Sailing along AB, then AH = amount north it goes.

N 50° E.

$\phi = 50°$.

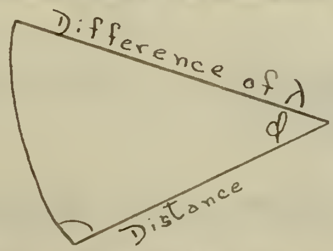
Sailing only east or west, ϕ always expresses latitude, while λ expresses difference of longitude.

AB sec ϕ = A'B'.

d. sec ϕ = difference of λ .

Introduce new triangle (artificial).

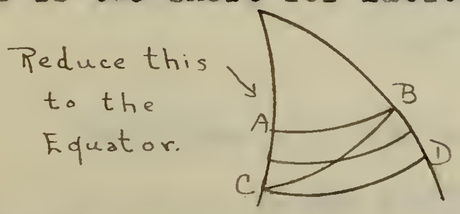
Fig. 20.


Difference of λ
ϕ
Distance

$$\sec \phi = \frac{\text{diff. of } \lambda}{\text{dist.}}$$

Plane sailing triangle or parallel sailing triangle

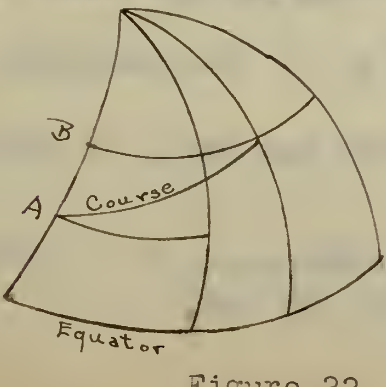
AB is too short for latitude, CD too long.


Reduce this to the Equator.
A
B
C
D

∴ take latitude of point midway. Sail along CB.

Figure 21.

Use the departure in dealing with this.


B
A
Course
Equator

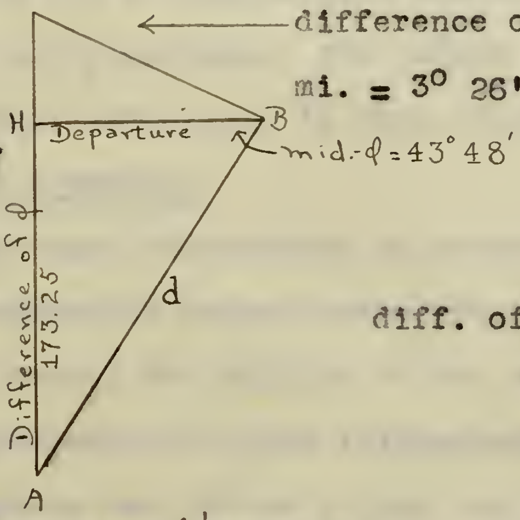
The latitude at which we start is A, the latitude at the end is B.

Figure 22.

Lay off circles or arcs, equal to the departure line and draw meridians. They will not coincide at all. Therefore we ought to get this arc at the equator.

Assume this to work if we put the latitude halfway between the two. This is called middle state sailing and we can get both the longitude and the latitude.

Figure 23.



difference of λ = 206 mi. = 3° 26'

b = bearing or angle.

b = N 40° E.

d = 225 mi.

diff. of ϕ = d. cos b.
= 225 (.77)
= 173.25 mi.

Started at A at

ϕ	42°21'N
λ	71°19'E

Increasing latitude by 173.25 mi., or 2° 53.25 minutes.

Arrived at

ϕ	45° 14.25' N
λ	67° 53' E

Departure = d. sin b
= 225 (.64)
= 144.00

Difference of λ = d.sec ϕ

$\therefore$ Diff. λ = 144 sec 43° 48'

$$\text{Or Diff. } \lambda = \therefore \frac{144}{\cos 43^\circ 48'} = \frac{144}{.7} = 205.9$$

$(\cos 43^\circ 48' = 0.69214)$

or diff. $\lambda = 206$ mi.

Change to degrees, minutes, and seconds. 206 mi. equals 3°26'. Subtract 3°26' from 71°19' and arrive at 67° 53' E.

In using this the navigator adds correction to mid ϕ and is therefore more accurate. The better way to do this is called Mercators' method and is more accurate.

C. Nautical Astronomy

This uses observation of heavenly bodies. This deals with spherical trigonometry but a simple explanation is not beyond the ability of the high school student with a knowledge of plane trigonometry. We treat the earth as a sphere and not as a flat surface as in the first two cases. Suppose we are concerned with distances from Boston to the Cape Verdi Islands. Hold a string and pull it taut to get the shortest distance, which is measured on a great circle of a sphere. Take the meridians from the North Pole to Boston and from the North Pole to the Cape Verdi Islands and the arc between them and we have a spherical triangle.

90°-lat. = co-lat. = distance from B to N.

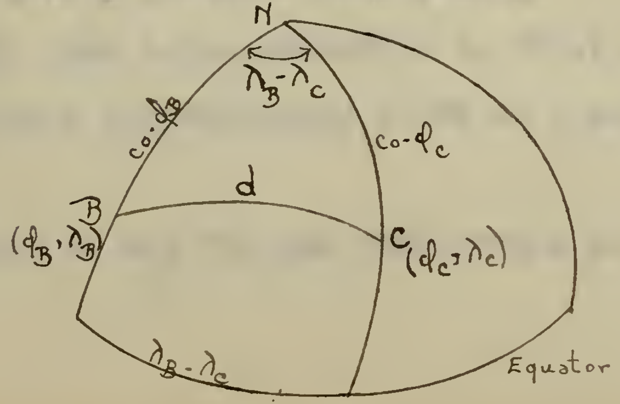


Figure 24.

Prime meridian.

Note Spread planes apart and therefore change the angles, and the arc of the great circle is measured and also the angle at the center, or the tangent at the pole. Therefore we have three ways to measure the angle and therefore the angle above is $\lambda_B - \lambda_C$.

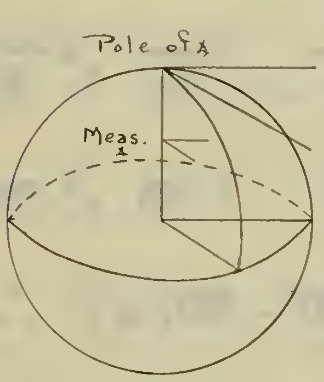


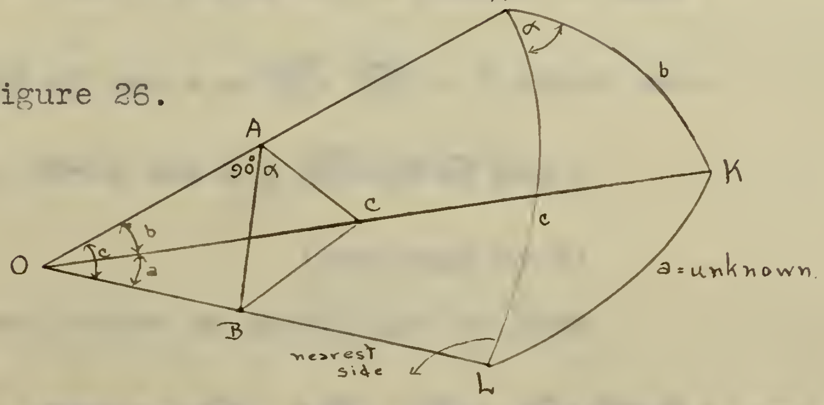
Figure 25.

The problem then becomes such that 'd' is the unknown in the spherical triangle in which we know two sides and the included angle.

Take a new figure so that we can get all of this in. Draw radii from the three vertices.

α = angle known and c and b are the known sides, while a = the distance.

Figure 26.



In the plane of OHL, draw a perpendicular to OH at A. In the plane of OHK draw a perpendicular to OH at A and this angle BAC = α

Angle KOL is measured by arc 'a' and therefore = a.

From plane trigonometry we get the square of the side opposite an acute angle.

1. $\overline{CB}^2 = \overline{AB}^2 + \overline{AC}^2 - 2AB \cdot AC \cdot \cos \alpha$ (from $\triangle$ ABC)

2. $\overline{CB}^2 = \overline{OB}^2 + \overline{OC}^2 - 2OB \cdot OC \cdot \cos a$ (from $\triangle$ OBC)

$$\therefore \quad 0 = (\overline{OB}^2 - \overline{AB}^2) + (\overline{OC}^2 - \overline{AC}^2) + 2\, AB \cdot AC \cdot \cos \alpha$$
$$- 2\, OB \cdot OC \cos a$$

(Subtracting equation 1 from 2)

$$\therefore \quad 2\, OB \cdot OC \cdot \cos a = (\overline{OB}^2 - \overline{AB}^2) + (\overline{OC}^2 - \overline{AC}^2)$$
$$+ 2\, AB \cdot AC \cos \alpha$$

(Transposing)

But $\overline{OB}^2 - \overline{AB}^2 = \overline{OA}^2$ (from $\triangle$ OAB)

And $\overline{OC}^2 - \overline{AC}^2 = \overline{OA}^2$ (from $\triangle$ OAC)

Substituting these values we have

$$2\, OB \cdot OC \cos a = \overline{OA}^2 + \overline{OA}^2 + 2\, AB \cdot AC \cos \alpha$$

$$\therefore \quad OB \cdot OC \cos a = \overline{OA}^2 + \overline{AB} \cdot \overline{AC} \cos \alpha$$

(dividing by 2)

Now divide by OB·OC, and we have

$$\therefore \cos a = \frac{OA}{OB} \cdot \frac{OA}{OC} + \frac{AB}{OB} \cdot \frac{AC}{OC} \cos \alpha$$

Since $\frac{OA}{OC} = \cos b$; $\frac{OA}{OB} = \cos c$

And $\frac{AB}{OB} = \sin c$; $\frac{AC}{OC} = \sin b$

$$\therefore \cos a = \cos b . \cos c + \sin b . \sin c . \cos \alpha$$

This can be used for many purposes in spherical trigonometry. Get "a" in degrees, minutes, and seconds, change to minutes and hence to miles.

Suppose a navigator is going to sail on a great circle--this is called great circle sailing. Very serious difficulties are encountered in this because the course is changing constantly on any great circle. He sails, therefore, on "rhumb" lines which are very short lines to keep close to the great circle but not on it.

In composite sailing we sail first on a great circle up to a certain parallel, usually determined before the boat sails, along the parallel, then sail on another great circle to port. This is used in the North Atlantic because the great circle passes through ice fields and in the Pacific because it must pass through islands. The great circle distance would use the formula derived above and then go back to the parallel.

Suppose we are making a voyage from San Francisco to Yokohama.

San Francisco	(37° 47'.0 N
	(122°32'.0 W
Yokohama	(35° 27' 2 N
	(139°40'.4 E

Considering the three types of voyages the trips would have been as follows:

1. Great circle voyage -- 4,467 miles.
2. Composite sailing, on 42nd parallel of 42 N -- 4,532 miles. (Length of path on parallel is 1,315 miles.)
3. Parallel sailing -- 4,920 miles.

Let us consider the composite sailing. We use the great circle that strikes the parallel at tangency--one from San Francisco and the other from Yokohama.

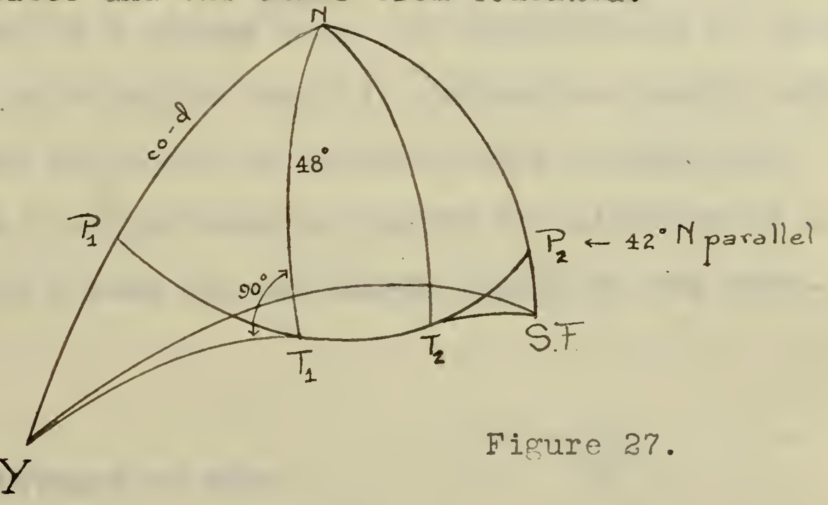


Figure 27.

$NY = co\text{-}\phi$

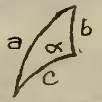
$NT_1 = 48^\circ$

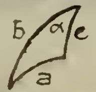
$P_1\ P_2$ = small circle at right angles to the meridians.

Angle $YT_1N = 90^\circ$.

$\therefore$ We have a right spherical triangle NYT_1.

$$\cos a = \cos b \cdot \cos a + \sin b \cdot \sin c . \cos \alpha$$

Here suppose that angle $\alpha = 90^\circ$. 

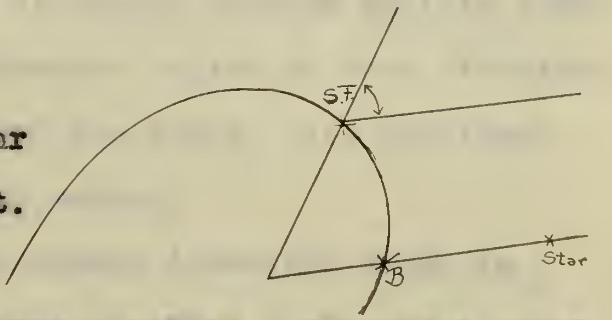
And we have $\cos a = \cos b . \cos c$ and therefore we know a and b and can get c. If you have c, just change lettering of figure and forget the right angle. 

Therefore, we can get YT_1 and angle YNT_1 and in the same way get T_2F and

angle FNT_2 , then subtract and get the difference of longitude and hence the distance on the parallel T_1T_2 .

In general, too much astronomy is required in nautical astronomy in determining the positions of astronomical bodies. The line of position method is not routine but it is most powerful.

A rather interesting story is told of the use of this method. On December 17, 1837, Capt. T. H. Summer of Boston was sailing in a stormy sea. No observations had been made for many days but he knew his reckonings fairly well. He was hard off the coast of Ireland, near a rocky lea shore. He got a single observation of the altitude of a star. How does a star in our zenith appear to San Francisco?



Zenith distance of star is measured by sextant.

Figure 28.

Two lines are practically parallel. Suppose a point in South America as far from Boston as San Francisco and get the zenith distance. We find it the same.

Boston just now is called sub-stellar and changes from minute to minute. Take a star in zenith of Boston and we have a cone with its vertex at the center of the globe. All points on the circle are at the same zenith distance.

It is obvious then that the zenith distance of any

star at any moment will be the same at all points of any circle on the surface of the earth which has the sub-stellar point for a center or pole.

Sumner reckoned the zenith distance of the star he had observed and located on a chart the substellar point of this star. He also got the direction of the star by the compass. He drew a line in the opposite direction; that is, if SS were SE, he drew a NW line. The zenith distance of the star gave him the length of the line and he changed it to nautical miles. He drew lines at right angles and knew he was somewhere on this line. He extended this line and found it passed through Small's Light on the Irish coast. He therefore sailed in that direction and within an hour he reached the light. All civilized nations today use the Sumner method.

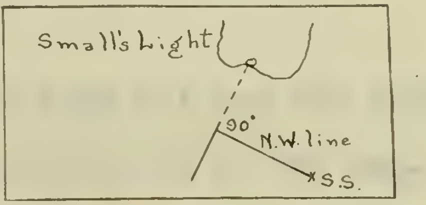


Figure 29.

The radio compass is a modern invention which is playing an important part in navigation in locating a ship's position, especially when it is near land. It is essentially a powerful receiving set with a large loop aërial, such that, by turning the loop, radio signals of varying intensity are obtained. In plotting a ship's position from such signals, a navigator uses straight lines on his chart and not arcs of great circles because of the relatively small distances involved.

Suppose, for example, that a vessel is fog bound at

point P and that X and Y are radio transmitting stations on the coast. The line of direction PX' may be found by rotating the needle and loop of the radio compass until signals from the station are not heard. The loop is then at right angles to the line from that station and the needle of the compass is pointing directly towards the station. (1)

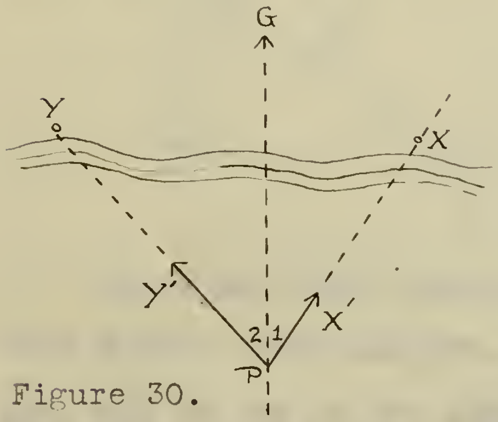


Figure 30.

This line PX' and the north-south line PG of the magnetic compass determine the angle or bearing (∡ 1) of the ship with reference to X. In like manner, ∡ 2 determines the bearing with reference to station Y. The navigator may now make a chart and with XG' as a north-south line through X draw ∡ 3 = ∡ 1. In like manner, he draws ∡ 4 = ∡ 2.

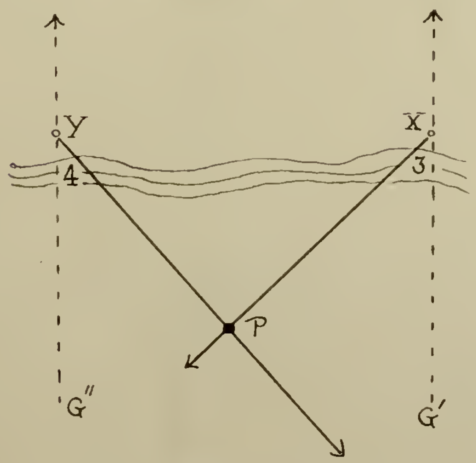


Figure 31.

The intersection of XP and YP determines the position of the ship on the chart. In this method, the radio compass is on the ship and the station at X and Y serve as radio lighthouses. In other cases, the radio operator on a vessel calls

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(1) Strader, W. W. & Rhoads, L.D. - Solid Geometry - John C. Winston & Co. - pages 148 - 149.

station X, which is equipped with a radio compass, and station X gives the vessel its bearings. The operator, in like manner, obtains the bearings from stations Y and Z. The lines which are then drawn on a chart may not all pass through the same point, but form a triangle called by navigators a "cocked hat". The center of the "cocked hat" is taken as the position of the vessel.

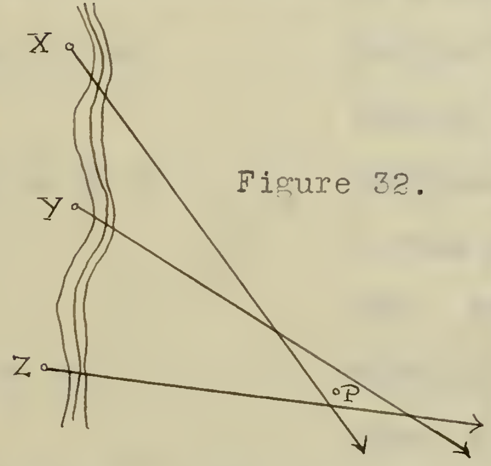

Figure 32.

There are other interesting and valuable uses of this modern invention but, to whatever use the radio compass may be put in the future, a knowledge of geometry will be essential.

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APPRECIATION UNIT V.

THE MATHEMATICS USED IN DETERMINING THE VELOCITY OF LIGHT.

THE VELOCITY OF LIGHT.

Did you ever notice how sunlight comes through a small opening into a dark room? Does it not come in a straight line, illuminating only those dust particles which are in that line? When no other force interferes, light seems to pass from the source to the eye in straight lines. This at once suggests inquiry whether or not the eye instantly experiences the sensation of light when a candle is uncovered.

The earliest attempt to solve this question was (1) made by the Florentine Academy after a method proposed by Galileo. A light on an eminence was uncovered and flashed to a station on a distant hill where a second observer also having a covered light was watching. As soon as the flash was seen by the second observer he uncovered his light, sending an answering flash back to the first station. The idea was, that the time which elapsed between the first observer's uncovering his own lamp and his seeing the second lamp would be equal to the time taken by the light to go from him to the second observer and back. The method failed owing to the enormous velocity of light, the time taken by it to travel the distance in question being very much less than the time necessary to uncover one lamp or to see another. The time was not so short, however, that mathematics, always exact and precise, could not detect it.

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(1) Kimball, A.L. - College Physics - Henry Holt & Co. page 548.

The velocity of light can be determined experimentally by four separate methods. These, taken in order of their discovery, are Römer's method, the aberration method, Fizeau's method and Foucault's method. Suppose we investigate these methods and see how mathematics again played the leading role as the "handmaid of the sciences".

I. Römer's Method

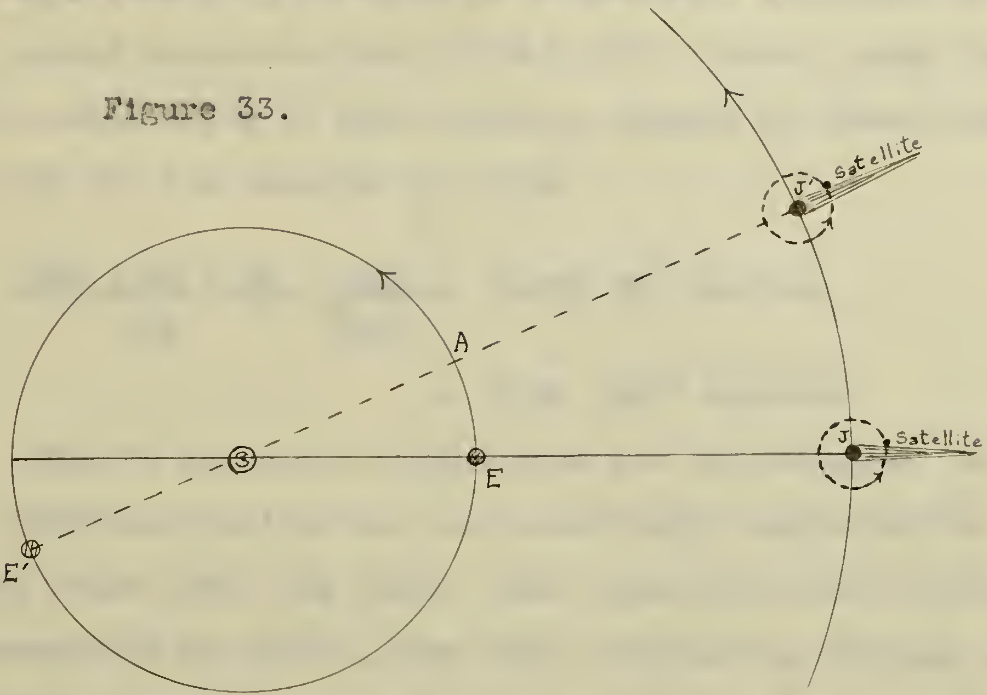
Four of the moons of the planet Jupiter are large enough to be observed by an ordinary telescope. They are, of course, dark bodies and are illuminated solely by the reflected light of the sun; consequently when they enter the shadow cast by Jupiter they are eclipsed or disappear. If we assume that they rotate about Jupiter with uniform angular velocity, the interval of time that elapses between the successive eclipses of any one satellite should always be the same. Römer observed a peculiar variation in the times of occurrence of the eclipses.

When the earth was approaching Jupiter they occurred too close together, and, when the distance between the earth and Jupiter was increasing, they occurred too far apart. He explained the difference by means of the time taken by light to pass through space.

In Figure 33 below, let S represent the sun, and let the two circles be the orbits of the earth and Jupiter. E and J are the positions of the earth and Jupiter when they are in conjunction, i.e. nearest one another; and E' and J' their positions when in opposition, i.e. far-

thest from one another. Jupiter takes 11.86 years to make one revolution round the sun, so that it moves only from J to J', while the earth moves from E to E'.

Figure 33.



Römer, a young Danish astronomer, observed accurately the instant at which a satellite of Jupiter passed into Jupiter's shadow when the earth was at E, and he forecast from the known mean time between such eclipses, the exact instant at which a given eclipse ought to occur six months later, when the earth should be at E'. It actually took place 16 minutes 36 seconds (or 996 seconds) later. Römer concluded that the 996 seconds' delay represented the time required for light to travel across the earth's orbit, i.e. from A to E'. This distance, the diameter of the earth's orbit, was known only approximately as 186,000,000 miles. Hence, Römer's result was slightly different from the most recent results. To determine the diameter of the earth's

orbit he made use of the solar parallax, the angle subtended by the earth's radius at the sun. This is one of the most difficult problems of astronomy. According to the latest determinations it is 8.80". Hence, using the more recent value of 1002 seconds, instead of Römer's 996, we find for the velocity of light

$$\frac{360 \times 60 \times 60}{8.8} \cdot \frac{3963}{1002} = 1.855\ 10^{5} \text{ mls/sec.} \qquad (1)$$

$$= 2.98\ 10^{10} \text{ cms/sec.}$$

Römer's ingenious explanation was too advanced for (2) his fellow-scientists and was accordingly neglected for fifty years after his death. His determinations, slightly corrected in detail, have been verified in general, so that the figures given by Michelson are 186,360 miles per second.

II. The Aberration Method.

The apparent direction of the light from a star depends on the motion of the telescope. For example, in Figure 34, let OS be the true direction of a star, let the telescope be pointed in the direction OS, and let the telescope and observer be moving with velocity v in the direction OP. Then, when the light is passing down the

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(1) Houstown, R. A. - A Treatise on Light- Longman's Green & Co. - page 120.

(2) Fuller, Brownlee & Baker - Elementary Principles of Physics - Allyn & Bacon Co. page 322.

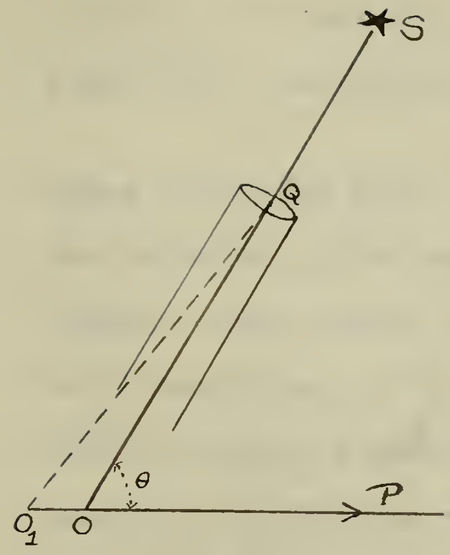

Figure 34.

telescope, the latter is moving sideways; consequently the path of the central ray relative to the telescope is shown by the dotted line QO_1 , and, if the image of the star is to appear in the middle of the field on the cross-wires, the telescope must be pointed in the direction O_1Q. Let $\measuredangle\theta$ be the true direction of the star, let V be the velocity of light, and let t be the time taken by the light to travel down the telescope. Then $O_1Q = Vt$ approximately, since OO_1 is small, and $O_1O = vt$. In the $\triangle\ QOO_1$ (1)

$$\frac{\sin O_1QO}{O_1O} = \frac{\sin QOO_1}{QO_1}$$

By substitution, we have

$$\frac{\sin O_1QO}{vt} = \frac{\sin\ \theta}{Vt}$$

Or, since $\sin O_1QO$ is very small

$$\measuredangle\ O_1QO = \frac{v}{V}\ \sin\theta$$

Thus, owing to the motion of the telescope, the star is displaced in the direction of that motion in front of its true position by an angle equal to $\dfrac{v \sin\ \theta}{V}$

The earth moves in its orbit about the sun with a

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(1) By the Law of Sines -- See any Plane Trigonometry.

velocity of about $18\frac{1}{2}$ mls/sec. If this value be substituted for v and sin θ be put equal to unity, $\frac{v \sin \theta}{V}$ takes the value 20". Thus if the telescope is moving with the velocity of the earth, the stars receive an apparent angular displacement varying from 20" to 0' according to their positions in the heavens. This apparent displacement is known as aberration. Its effect is to make the apparent position of each star execute an annual motion about its true position. The value of the aberration constant adopted at present as a result of observations is 20.492".

Now, the mean velocity of the earth in its orbit is 18.51 miles per second, and we may calculate the velocity of light V from the relation

$$\tan (20.492") = \frac{18.51}{V}$$

which gives 186,400 miles or 299,930 kilometers per second.

III. Fizeau's Method.

The first terrestrial method of determining the velocity of light was carried out by Fizeau in 1849. His arrangement is shown in Figure 35.

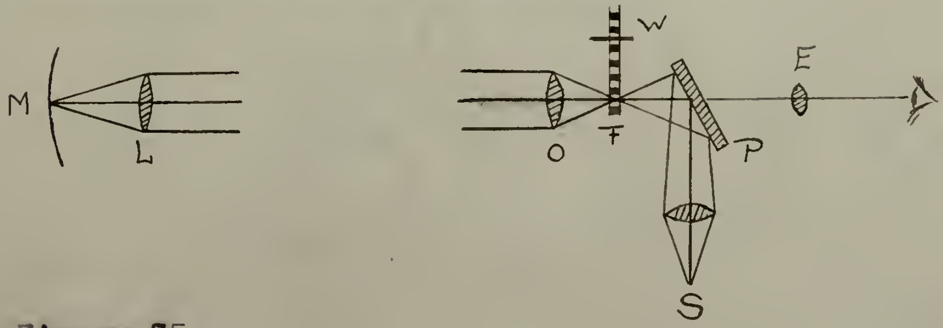

Figure 35.

A beam of light from a source S passes through a converging lens system, is reflected by a glass plate P, and comes to a focus at F. It is then made parallel by the lens O, falls on the lens L and is brought to a focus on the surface of the concave mirror M. The radius of curvature of this mirror is equal to ML, its distance from the lens; the central ray of any pencil through the lens thus falls on the mirror normally and is reflected back the way it comes, even though inclined to the axis of the mirror. The lens and mirror L and M thus direct the beam back through the lens O to form a real image once more at F, and the observer looks at this real image through the eyepiece E and the glass plate P. W is a toothed wheel, and, as it rotates, its teeth pass one after another through the point F alternately stopping and letting through the light. If the notches and teeth in the wheel are of equal width a tooth moves forward just its own width in the time that light requires to go to the distant station and back, light which must have passed out through an opening will on returning find the opening closed and will cut off from the observer. If the speed is then doubled, light passing out through one opening will return through the next one; at a still higher speed it will be eclipsed again. It is therefore only necessary to observe the speeds at which the light is completely eclipsed to be able to determine the velocity of light, when the distance between the two stations and the number of teeth in the

wheel are known. In Fizeau's apparatus there were 720 teeth in the wheel and the first eclipse was noticed when the wheel made 12.6 revolutions per second. Therefore the time required for light to travel twice the distance between the two stations was only

$$\frac{1}{720 \times 12.6 \times 2} = \frac{1}{18143} \quad \text{seconds.}$$

The stations were 8.633 kilometers apart, making the velocity of light 313,000 kilometers per second.

Perhaps a little more accurate method would be to determine (1) the angular velocity of the wheel for which the image disappeared. Let it be ω radians per second for the "n" th disappearance, and let V be the velocity of light in kms/sec. Then

$$\frac{2 \times 8.6}{V} = \frac{(2n-1)}{2} \cdot \frac{2\pi}{720} \cdot \frac{1}{\omega} ,$$

$$V = \frac{720 \times 2 \times 8.633}{(2n-1)\pi}$$

whence V can be calculated.

Using this same method, but with a distance of 23 kilometers, Cornu determined V = 304,000 kilometers per second or $3.004 \ 10^{10}$ cms/sec.

Perrotin used Fizeau's apparatus as modified by Cornu using a distance of 40 kilometers. He obtained $2.9986 \ 10^{10}$ centimeters per second, or 186,345 miles per second.

(1) Houstoun, R. A. - A Treatise on Light - Longman's Green Co. page 123.

IV. Foucault's Method.

Figure 36 shows the details of the method by which Foucault made a determination in 1862. It requires a much shorter distance than Fizeau's method. S is a rectangular aperture illuminated with light, Q is a plane parallel plate of glass, L a lens. R a plane mirror, which can be rotated about an axis perpendicular to the plane of the figure, and M a concave mirror.

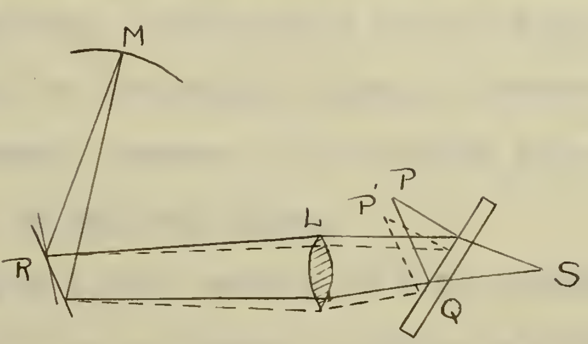

Figure 36.

The lens L forms an image of S on M. The center of curvature of M is at the center of R, hence, no matter what the position of R is, if the light from it falls on M at all it falls on it normally and is thus reflected back along its path. If R is rotating rapidly, it has moved through an appreciable angle by the time the beam returns from M, and consequently the light is reflected in the direction of the dotted rays to form an image at P'.

In Foucault's experiment the distance RM was 20 meters and the displacement PP' 0.7 mm. From the displacement the angle turned through by R was calculated

and then, the angular velocity of R having been determined, the time taken by the light to go from R to M and back was known and thus as in the other cases the velocity can be determined.

Foucault's method has been used and considerably improved by Michelson and Newcomb. Michelson, of the University of Chicago, placed the lens between R and M and was able to increase the distance RM to 600 metres. His result (1882) was 2.9985 10^{10} cms/sec. and Newcomb's result 2.9986 10^{10} cms/sec. Michelson's value of 299,860 kilometers per second is generally taken as correct. It is sufficiently correct, however, to remember it as 300,000 kilometers, or 186,000 miles. (1)

The mind can form little notion of this enormous speed. If light could be reflected by mirrors rightly placed, it would go 7½ times around the earth in one second. It is so small in comparison with interstellar distances, however, that the light which is now reaching the earth from the nearest fixed star, Alpha Centauri, started 4.3 years ago, while in the case of the more remote stars, light emitted thousands of years ago has not yet reached us! The light year has become the regular unit of interstellar distances. (2)

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(1) Houstoun, P. A. - A Treatise on Light - Longman's Green & Co. page 134.

(2) Millikan, R. A. & Gale, H. G. Elements of Physics Ginn & Co. page 385.

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APPRECIATION UNIT VI

THE ELECTRO-MAGNETIC THEORY OF LIGHT
AS A
PRODUCT OF MATHEMATICAL CONSIDERATION

THE ELECTRO-MAGNETIC THEORY OF LIGHT.

The Electro-magnetic Theory of Light was a shock to the sensibilities of some of the more conservative physicists, who were reluctant to abandon the 'elastic solid' ether which had served them so long and faithfully. In 1865, Clerk Maxwell published a memoir in which he proved mathematically that the same medium required for the Wave Theory of Light should also serve for the explanation of electro-magnetic effects. For some years, however, there was no experimental evidence for the existence of electro-magnetic waves. But Maxwell's investigations brought about a great simplification in electrical theory. In his memoir, Maxwell treated electrical phenomena from the dynamical point of view. Thus, in his theory, we find the expressions 'energy' of an electric current, 'elasticity' of the medium, and 'momentum' of the varying electro-magnetic field. Suppose we examine his theory and simplify it as much as we can.

We know that when a circuit is broken, the current does not instantly cease, but persists for a very short time. In other words, an electric current, like a material body, requires a good push to start it moving, but when once the inertia has been overcome and it has acquired momentum, it resists being stopped.

Maxwell, in considering the effects of self-induction, was thus led to think of an electric current as having kinship to the momentum of a moving body. The heating

effect of a current and the definite relationship between heat and work had made physicists familiar with the notion of electrical energy. Maxwell, however, went farther. He broke down the barrier between electro-static and electro-kinetic phenomena. He pictured all the phenomena of an electric current as those of a moving system, the motion being transmitted from one part of the system to another by forces subject to theuse of dynamical laws.

You have perhaps heard of the famous Maxwellian equations. In the mathematical part of his memoir he developed twenty equations in twenty variables. Let us note the following: (1)

a. The relation between electric displacement, true conduction and the total current compounded of both.

b. The relation between the lines of magnetic force and the inductive coefficients of a circuit, as already deduced from the laws of induction.

c. The relation between the strength of a current and its magnetic effects, according to the electro-magnetic system of measurement.

d. The value of the electromotive force in a body, as arising from the motion of the body in the field, the alternations of the field itself, and thevariations of electric potential from one part of the field to another.

e. The relation between an electric current and the electromotive force which produces it.

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(1) Turner, D. M. - Makers of Science - Oxford Univ. Press - page 134.

f. The relation between the amount of free electricity at any point, and the electric displacement in the neighborhood.

By combining these equations Maxwell obtained a simple expression for the repulsion between two electrified bodies in electro-dynamic units.

If ε_1 and ε_2 are the charges, and r the distance between them, the force was given by the formula

$$(1) \qquad F = \frac{K}{4\pi} \cdot \frac{\varepsilon_1 \varepsilon_2}{r^2},$$

K being the specific conductive capacity of the medium. But from the law of inverse squares

$$(2) \qquad F = \frac{N_1 \quad N_2}{r^2},$$

where N_1 and N_2 are the charges in electrostatic units.

Now, if "v" represent the ratio between the electro-magnetic and electro-static units, it follows that

$$N_1 = v \cdot \varepsilon_1,$$

And $$N_2 = v \cdot \varepsilon_2,$$

(3) Whence $$F = \frac{v^2 \varepsilon_1 \varepsilon_2}{r^2}$$ (substituting in (2))

Therefore $$\frac{v^2 \varepsilon_1 \varepsilon_2}{r^2} = \frac{K}{4\pi} \cdot \frac{\varepsilon_1 \varepsilon_2}{r^2}$$

(substituting equals for equals)

Then $v^2 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot 4\pi\, r^2 = K \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot r^2$ (clearing of fractions)

Therefore $4\pi\, v^2 = K$ (dividing by $r^2 \cdot \varepsilon_1 \cdot \varepsilon_2$)

And $v^2 = \dfrac{K}{4\pi}$

Then $v = \pm \sqrt{\dfrac{K}{4\pi}}$

This ratio between the units had been found from the experiments of Weber, Kohlrausch and others to be numerically equal to the velocity of light.

We have seen how Maxwell in his investigation of electromagnetic phenomena sought an interpretation in terms of the properties of the insulating medium. In the Wave Theory of Light of course we must consider a medium, endowed with certain properties in order to interpret such phenomena as reflection and refraction. Maxwell deemed it unphilosophical to fill space with a different kind of medium each time a new phenomenon is explained. It seemed reasonable then that the same kind of medium required for the Wave Theory of Light should also serve for the explanation of electro-magnetic effects.

He turned once more to his equations. Assuming that an electro-magnetic disturbance is propagated through the field by means of a plane wave, that is, a wave in which the electric intensity is the same at any instant over the whole plane, he obtained a simple expression for the

velocity of such a wave. If V denotes the velocity of the wave, K the specific inductive capacity as before, and μ the permeability of the medium, Maxwell found that

$$V = \pm \sqrt{\frac{K}{4\pi\mu}} ;$$

but we have already shown that

$$v = \pm \sqrt{\frac{K}{4\pi}} ,$$

when v is the ratio of the electro-magnetic to the electrostatic units.

Now, since the only medium in which experiments have been made for values of K is air, for which $\mu = 1$, he could write

$$V = \sqrt{\frac{K}{4\pi}}$$

Hence $V = v$

i.e. the velocity of propagation of the wave is equal to the ratio v between the units, and this by experiment is numerically equal to that of light.

Thus Maxwell by purely theoretical investigations deduced that an electro-magnetic disturbance travels with the velocity of light. What more reasonable inference than that light itself is an electro-magnetic phenomenon? Maxwell extended his theory, however, and showed that a periodic electrical disturbance, which in its turn gives rise to a periodic magnetic disturbance, should travel

with a wave motion the velocity of which should be the same as that of light. "During Maxwell's time it was (1) realised that the ratio of the electro-magnetic to the electro-static unit has the *dimensions* of a velocity.

This consideration lent some support to Maxwell's views from the purely theoretical side of the subject. But ten years after Maxwell's death electro-magnetic waves were actually detected, their velocity calculated, and the results of experiment found to agree with the *predictions* of his theory. The achievement by which mathematical analysis led to results identical with those deduced from observations, and the result thus deduced found to be identical with results in an entirely different department, is surely among the most remarkable in the whole history of science."

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(1) Turner, D. M. - Makers of Science - Oxford Univ. Press - page 136.

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APPRECIATION UNIT VII

THE MATHEMATICS USED IN HETERODYNE RECEPTION.

Note: Units VII and VIII, involving some elementary calculus, seem to be better suited to the more advanced pupils. Both of these units, however, have been used with success to stimulate interest in third and fourth year classes in secondary mathematics.

THE MATHEMATICS USED IN HETERODYNE RECEPTION.

In the rapid growth of radio communication the appliances and methods used have undergone frequent and radical changes. In this growth progress has been made largely by new inventions and applications, and comparatively little attention paid to refinements of measurement.(1) The methods, formulas, and data used in radio work can not be properly understood or effectively used without a knowledge of the principles on which they are based. Since modern practice tends toward the exclusive use of continuous and undamped waves, sources which furnish undamped currents are coming into more general use than those which supply damped oscillations. It is only within the past few years that an almost ideal source has become available—namely, electron tube generators, such as the pliotron, audion, and oscillion. These electron tubes consist of an evacuated bulb containing the following three elements:

(a) A heated filament which acts as a source of electrons.

(b) A metal "plate" placed near the electron source. (Across the plate and filament, outside of the bulb, is connected a battery, so that an electron current flows from filament to plate).

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(1) Dellinger, J. H. - Radio Instruments and Measurements Circular #74 - Bureau of Standards Washington, D. C. page 5.

(c) A "grid" consisting of fine wire or of a perforated metal sheet placed between plate and filament so that the electrons have to pass through the grid to get from filament to plate. (1)

Using these sources, undamped high-frequency current can be obtained which is as steady as the current from a storage battery and strictly constant in frequency.

On account of the extreme importance of the three-electrode tubes, both as generators and as detectors, and since the full realization of their utility and a satisfactory explanation of their functions are of recent date, it is worth while to outline rather fully the phenomena upon which their operation depends.

There are two general fields in which the oscillating vacuum tube is used in radio, that is, as a source of high-frequency power for a continuous-wave transmitting station, and as a necessary part of any station receiving continuous-wave signals, by means of the heterodyne or "beat" method.

In continuous-wave telegraphy the signal received by the antenna does not have variations in amplitude; a dot, for example, might consist of 5000 cycles of a 50,000-cycle current, the amplitude of the current being constant for the duration of the 5000 cycles. The receiving circuit is excited by a small transmitter coupled and adjusted to produce oscillations differing very slightly

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(1) Dellinger, J. H. - Radio Instruments and Measurements
Circular #74 - Bureau of Standards
Washington, D. C. page 200.

in frequency from those of the incoming signal. If the input circuit of the detecting tube is continually excited by a locally generated frequency of 49,000 cycles, when the signal comes in the input circuit is excited by both 49,000 cycles and 50,000 cycles, the result being a high-frequency excitation the amplitude of which varies 1000 times a second. This high-frequency, variable amplitude, input voltage will give a 1000-cycle note in the telephones, connected in series with the plate circuit of the tube. In case the locally generated, high-frequency current is produced by the detecting tube itself, it is called *autodyne* reception; in case some device other than the detecting tube is used for impressing the local high-frequency current on the grid the scheme is called *heterodyne* reception.

When the magnetic flux through any circuit is changing, an electro-motive force is produced around the circuit, which lasts while the change is going on. The electro-motive force thus caused is called an induced electro-motive force, and the resulting current in the circuit is an induced current. The magnitude of the induced electromotive force at any instant is in every case equal to the rate of change of the magnetic flux through the circuit. This is expressed by the formula

$$e = \frac{d\phi}{dt}$$

where e represents the instantaneous value of the in-

duced emf in a circuit, and $\frac{d\phi}{dt}$ is an expression called the derivation of flux with respect to time and which tells the instantaneous rate of change of the flux. Although it is necessary to use derivatives in this discussion, it is believed that the treatment can, nevertheless, be understood by a person not familiar with calculus. To avoid the use of derivatives entirely would require such circumlocution that the treatment would be much less clear. This quantity, then, $\frac{d\phi}{dt}$ is the change of flux during a very small interval of time divided by the time, and its value may vary from instant to instant. If it remains constant for a certain length of time, then its value is the whole change of flux during that interval divided by the interval. To represent this induced electromotive force graphically, the curve has the mathematical form of a sine wave, which, on account of its simplicity, is assumed in most of alternating-current theory. It should not be forgotten, however, that sine-wave theory is in many practical cases only an approximation because the emf is not rigorously of sine-wave form.

Letting E_m = maximum electromotive force (1)

e = electromotive force at any instant

t = time

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(1) Dellinger, J. H. - Radio Instruments and Measurements Bureau of Standards-page 22.

ω = angular velocity

Then $e = E_m \sin \omega t$

This emf alternates in direction. Starting at a, Figure 37, it passes through a set of positive values, then through a set of negative values, and at b begins to repeat the same "cycle." In the time of one complete cycle, represented by the distance ab, the revolving radius OP makes one complete revolution, or passes through the angle 2π radians. The time required for one complete cycle being represented by T, it follows that

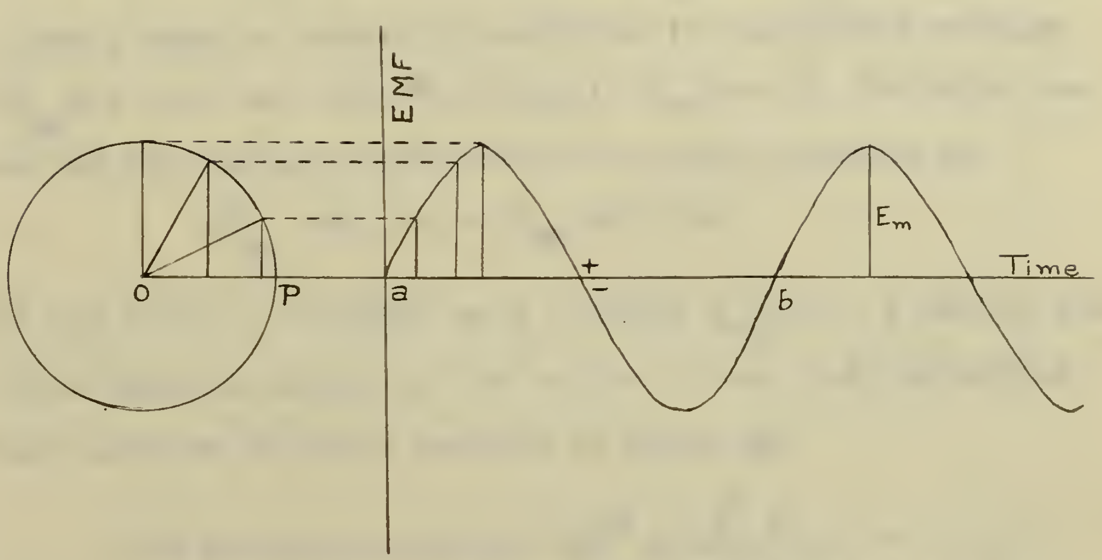

Figure 37. Sine wave developed by emf.

$$\omega = \frac{2\pi}{T}$$

The time T is called the "period" of the alternation. It is the reciprocal of the "frequency", which is the number of times per second that the electro-motive force passes through a complete cycle of values. It follows, denoting

frequency by f, that

$$\omega = 2\pi f$$

In considering the effect of frequency in electrical phenomena the quantity ω is found more convenient than f.

In heterodyne reception, the excitation of the input circuit when no signal is arriving is due to the voltage E^1_{mg} sin ω t, and when the signal, E_{mg} sin pt, is being received the actual excitation of the grid circuit is

$$E^1_{mg} \sin \omega t + E_{mg} \sin pt.$$

If the grid is actuated by a voltage E_{mg} sin ω t and if the (1) plate current varies as the square of the grid potential, the increase in plate current is given by

$$1/2 \, (\text{average value of } e_g)^2 \times \frac{d^2 I_p}{d \, E_g^2} \, .$$

Hence when the excitation is such as given by the curve below, the increase in plate current is

$$\Delta I_p = \text{average value of } \frac{(E^1_{mg} \sin \omega t + E_{mg} \sin pt)^2}{2} \, \frac{d^2 I_p}{d \, E_g^2}$$

$$= \left\{ \frac{E_{mg}^2}{4} + \frac{E^1{}_{mg}^2}{4} + \text{average value of } E_{mg} E^1_{mg} \sin \omega t \sin pt \right\} \frac{d^2 I_p}{d \, E_g^2}$$

The first two terms give the increase in the plate current which is constant, as long as the excitation is applied; their effect would produce an increase in the value of the plate current as read by a continuous-current

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(1) Morecroft, J. H. - Principles of Radio Communication
John Wiley & Sons - pages 484 - 485

ammeter in the plate circuit, but they would not produce a readable signal in the phones, giving only a slight click in the phones when the excitation is put on the grid and another when it is taken off.

Whatever audible signal is obtained must come from the third term; this may be written in the expanded form

$$1/2\ E_{mg}E^1_{mg}\ (\cos\ (\omega - p)\ t - \cos(\omega + p)t)\ .\ \frac{d^2\ I_p}{d\ E_g^2}$$

The average value of both these cosine terms is zero, but cos $(\omega - p)t$ may fluctuate so slowly as to produce an audible signal in the phones, and it is this term which is useful in continuous wave detection. The strength of signal is then measured by this term.

$$\Delta I_p \text{ (of audible frequency)} = \frac{E_{mg}E^1_{mg}}{2} \cos\ (\omega - p)t\ \frac{d^2\ I_p}{d\ E_g^2}$$

The frequency of this fluctuation in the plate current, which is the note heard in the phones, is adjustable by the operator, as he can make the value of ω anything he may desire. The ear and phone are both most sensitive at a frequency of about 800 cycles per second, so ω is generally adjusted to give $\frac{\omega - p}{2} = 800$

or $\frac{p - \omega}{2} = 800.$

It is to be noticed that whereas the response of the tube detector is proportional to the square of the signal strength for damped wave signals, it is proportional to

the first power of this signal strength when used for continuous-wave receiver. This fact makes the tube a better detector of signals for undamped, than for damped, waves, its sensitiveness not decreasing with the strength of signal so rapidly for one as it does for the other. The equation ΔI_p above shows also that the response to a given signal varies as $E^1{}_g$ does not change the value of

$$\frac{d^2 I_p}{d E_g^2} .$$

In this discussion, the symbols used have the following meanings:

I_p = effective value of alternating component of plate current.

E_g = effective value of alternating component of grid voltage.

E_p = effective value of alternating component of plate voltage.

E_m = the maximum value of voltage generated.

$\omega = 2 \pi f$, f being the frequency of the voltage.

t = time.

e = the value of voltage at any instant of time.

$e = E_m \sin \omega t = E_m \sin 2 \pi f t$ (sine wave of emf).

In order to use the vacuum tube as detector most efficiently it is necessary to have the amplitude of the voltage $E^1{}_g$ under control, and this can best be done by

using a separate tube for generating the voltage E^1_g, in addition to the detecting tube.

The operation of the heterodyne receiver is based on the idea of combining two currents of different frequencies to produce a resultant current of which the amplitude varies periodically, the frequency of this amplitude variation being the difference between the two component frequencies. If two sources, which separately furnish un- (1)

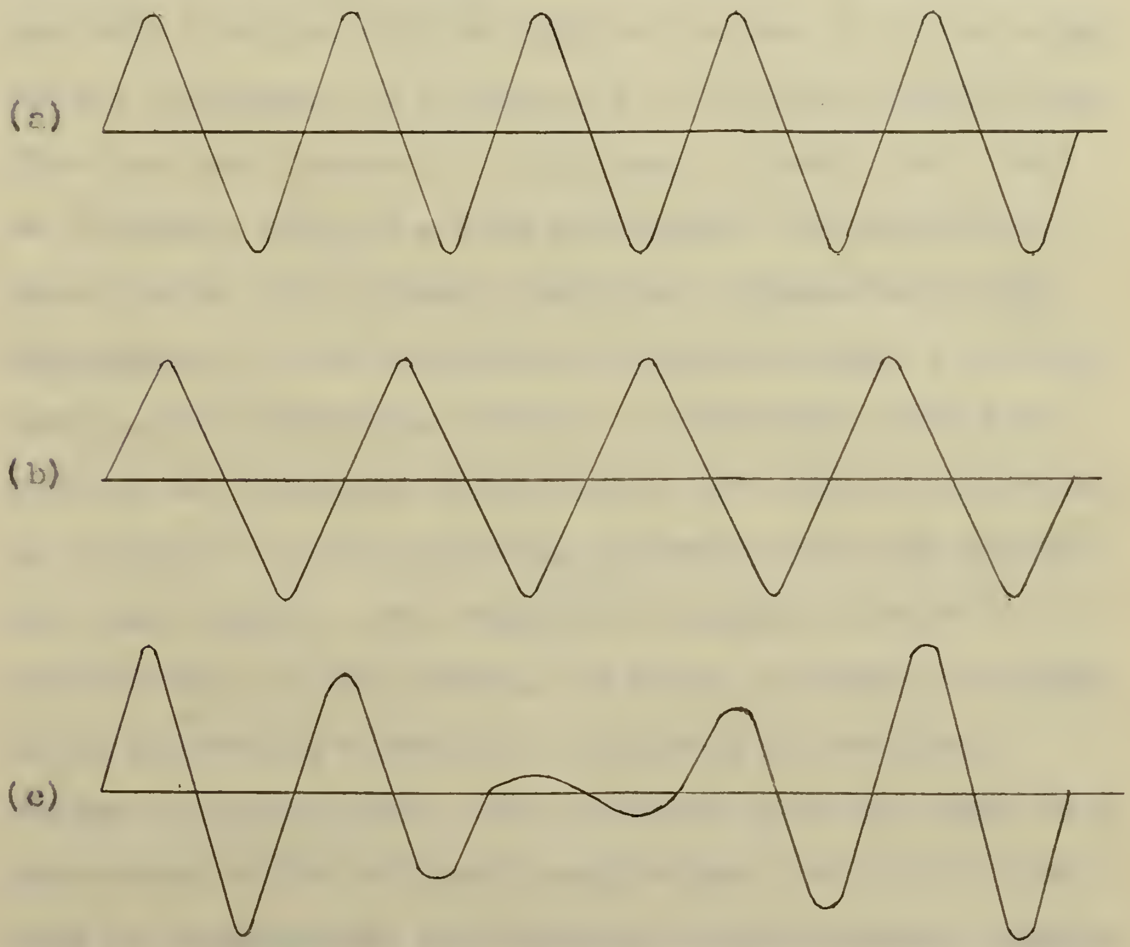

Figure 38. Principle of Heterodyne Reception.

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(1) Moyer, J. A. - Wostrel, J. F. - Practical Radio
MacGraw Hill Book Co. - page 135.

damped oscillations of, say, 100,000 and 101,000 frequency act together upon the same circuit, the resultant oscillations in the circuit, obtained by adding the components, will be of the form shown in (c) Figure 38, where (a) represents the incoming oscillations and (b) the oscillations produced by the tube. The amplitude of the combined oscillation will rise and fall, becoming a maximum when the component oscillations are in phase and a minimum when they are 180° out of phase. The beats or periodic rise and fall in amplitude occur at a rate equal to the difference in frequencies of the two oscillations. Thus the beat frequency in the case assumed above would be 101,000 - 100,000 = 1000 per second. If rectified, these beats will produce a note in a telephone of like frequency. In the reception of undamped signals by this method, the heterodyne method, the incoming signals represent one component oscillation. The other oscillation is generated in the receiving apparatus and both act in the same circuit. The rectified resultant furnishes a musical note in the phones, the pitch of which can readily be altered by varying the frequency of the local source of oscillations. The electron tube may serve as a convenient source of local oscillations and at the same time as an amplifier and detector of the received signals. This is called the autodyne or self-heterodyne method and is one of the most important of recent developments in the field of radio. Numerous circuits may be utilized to

produce these results, of which that shown in figure 39 may serve as an illustration. (1)

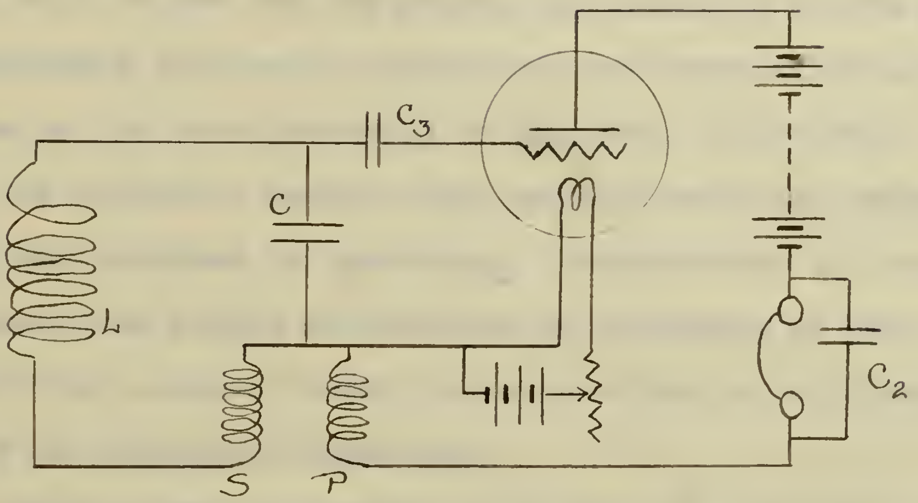

Figure 39. Use of electron tube as a regenerative amplifier.

Incoming signals set up oscillations in the antenna. By means of the coupling between the antenna and coil L oscillations of the same frequency are set up in the circuit LC, and are amplified on account of the feed back between S and P. Further, the coupling between S and P is such that the tube oscillates, the frequency of these oscillations depending largely upon the constants of the circuit LC. If this latter frequency is adjusted to be slightly different from that of the incoming oscillations, beats will result and the potential of the grid will follow the beat oscillations. In the case of reception with a grid condenser, there will be an increased flow of negative electricity from the filament to the grid when this latter is positive and its mean potential will be lowered.

(1) Dellinger, J. H. - Radio Instruments and Measurements Bureau of Standards - page 209.

Thus, as the oscillations in the beat are increasing the potential of the grid will become lower. The plate current will follow the variations in potential of the grid, reproducing the beat oscillations and decreasing in mean value as the mean potential of the grid is lowered.

By this beat method high sensitiveness and selectivity are attained in receiving. Interference is minimized because even slight differences in frequency of the waves from other sources result in notes either of different pitch or completely inaudible.

Still another important principle in valve reception is that of the 'super-heterodyne'. We have noted the difficulties involved in obtaining any considerable amplification of short waves of very high frequency. In this country and in Europe generally, the problem of the reception of waves of high frequency has been attacked by paying great attention to the elimination of undesired capacity effects between different parts of the apparatus, and by designing valves in which the capacities between the various electrodes are reduced to a minimum. Abroad, however, the problem has been tackled in another way, both by Levy and by Armstrong, who have developed the principle of the 'super-heterodyne'. In this method, the incoming signal of high frequency is heterodyned by a local oscillator in such a way that, instead of beats of audible frequency being produced, super-audible (or radio-frequency) beats of a predetermined frequency are obtained. The os-

cillations of this radio-frequency are passed through an amplifier designed for this particular frequency and then rectified. Let us suppose that the wave to be received is (1) 100 meters, corresponding to a frequency of 3,000,000 cycles per second, that it is undamped, and that an intermediate amplifier has been designed for maximum efficiency at 100,000 cycles. Then the original oscillation is heterodyned by local oscillations of 3,100,000 or 2,900,000 cycles per second. The beats resulting are of a frequency of 100,000 cycles; and these are rectified and applied to the 100,000-cycle amplifier and heterodyned again to produce beats of audible frequency, and finally detected before application to the telephones.

From this analysis we gain an insight into and an understanding of the fact that it is mathematics that makes these things possible. These "miracles of the air" are simple mathematical achievements.

(1) Brown, O. F. - The Elements of Radio Communication
Oxford University Press - page 153.

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APPRECIATION UNIT VIII.

THE USE OF COMPLEX NUMBERS IN THE THEORETICAL TREATMENT OF ALTERNATING CURRENTS

THE REPRESENTATION OF ALTERNATING CURRENTS BY COMPLEX QUANTITIES.

Electric current is the rate of flow of a quantity of electricity. The most familiar and most important properties of an electric current are (1) the heating effect produced in a conductor in which it flows, and (2) the magnetic field surrounding it. When a current flows continuously in the same direction, as the current from a battery, it is called a direct current. When the current periodically reverses in direction, it is an alternating current. The alternation of current is accompanied by a reversal of direction of the magnetic field around the current. On this account alternating currents behave very differently from direct currents. The uses of alternating currents may be divided, roughly, into three groups, separated according to the frequency of alternation of the current used.

Electric power applications, 20 to 100 per second.
Telephony, 100 to 20,000,000 per second.
Radio, 20,000 to 2,000,000 per second. (1)

The study of alternating current phenomena, and therefore also of electric oscillations, is assisted by the adoption of simple mathematical methods for representing the quantities with which we are concerned. The usual method of procedure is to express the instantaneous value of a periodic current or electro-motive force as a function

(1) Dellinger, J. H. - Radio Instrument & Measurements
Bureau of Standards - page 7.

of the maximum value of the current or force during the phase, and of the time expressed as a function of the complete periodic time. In actual practice, however, we have to distinguish between the instantaneous value, which varies constantly as a function of the time, and the integral value, which characterizes the wave as a whole.

As such integral value, almost exclusively the effective value is used, that is, the square root of the mean square. By the arithmetic mean, or average value, of a wave the arithmetical average of all the instantaneous values during one complete period is understood.

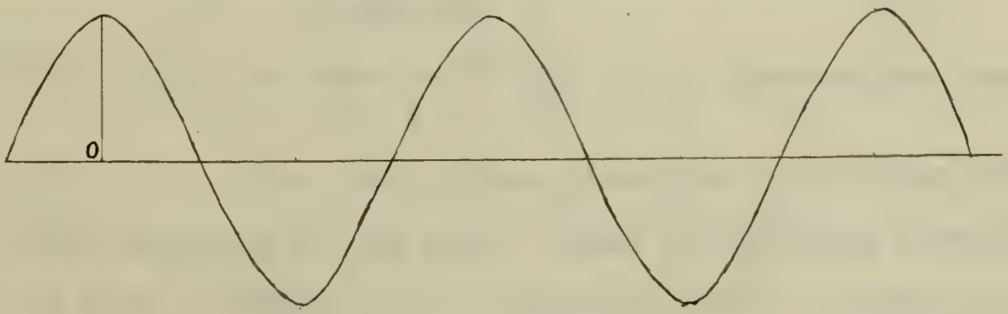

Figure 40. Alternating Wave.

This arithmetic mean is o, as in Figure 40, and the wave is called an alternating wave. Thus, an alternating wave is a wave whose positive values give the same sum total as the negative values; that is, whose two half-waves have in rectangular coördinates the same area, as shown in Figure 40.

In a sine wave, the relation of the mean to the maximum value is found in the following way:

Let, in Figure 41, AOB represent a quadrant of a cir-

cle with radius 1.

Then while the $\measuredangle \theta$ traverses the arc $\frac{\pi}{2}$ from A to B, the sine varies from 0 to OB = 1. Hence the average variation of the sine bears to that of the corresponding

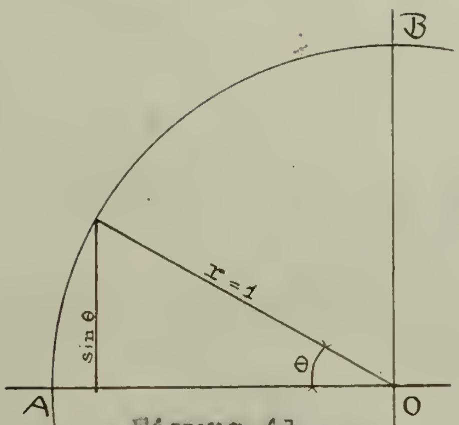

Figure 41.

arc the ratio $1 \div \frac{\pi}{2}$, or $\frac{2}{\pi} \div 1$. The maximum variation of the sine takes place about its zero value, where the sine is equal to the arc. Hence the maximum variation of the sine is equal to the variation of the corresponding arc, and consequently the maximum variation of the sine bears to its average variation the same ratio as the average variation of the arc to that of the sine, that is,

$1 \div \frac{2}{\pi}$, and therefore the mean value of sine wave $\div$ maximum value $= \frac{2}{\pi} \div 1 = 0.63663$.

Consequently, the only integral value of an alternating wave which is of practical importance, as directly connected with the mechanical system of units, is that value which represents the same power or effect as the periodical wave. This is called the effective value. Its square

is equal to the mean square of the periodic function, that is:

The effective value of an alternating wave, or the value representing the same effect as the periodically varying wave, is the square root of the mean square. (1)

In a sine wave, its relation to the maximum value is found in the following way:

Let, in Figure 42, AOB represent a quadrant of a circle with radius 1.

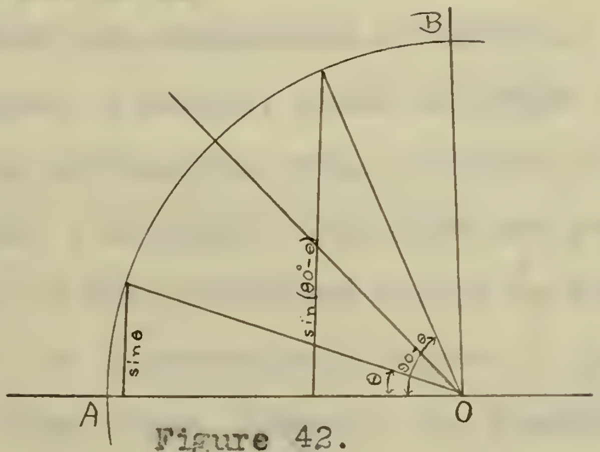

Figure 42.

Then, since the sine of any angle, θ , and its complementary angle, $90^{\circ} - \theta$, fulfill the condition,

$$\sin^2 \theta + \sin^2 (90 - \theta) = 1,$$

the sines in the quadrant, AOB, can be grouped into pairs, so that the sum of the squares of any pair = 1; or, in other words, the mean square of the sine = 1/2, and the square root of the mean square, or the effective value of the sine = $\frac{1}{\sqrt{2}}$. That is:

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(1) Steinmetz, C. P. - Theory & Calculations of Alternating Current Phenomena - MacGraw Hill Co. page 14.

The effective value of a sine function bears to its maximum value the ratio,

$$\frac{1}{\sqrt{2}} \div 1 = 0.70711$$

While alternating waves can be, and frequently are, represented graphically in rectangular coördinates, with the time as abscissae, and the instantaneous values of the wave as ordinates, the best insight with regard to the mutual relation of different alternating waves is given by their representation as vectors, in the so-called crank diagram. A vector, equal in length to the maximum value of the alternating wave, revolves at uniform speed so as to make a complete revolution per period, and the projections of this revolving vector on the horizontal then denote the instantaneous values of the wave. For numerical calculation, however, the graphical method is generally not well suited, owing to the widely different magnitudes of the alternating sine waves represented in the same diagram, which make an exact diagrammatic determination impossible. For numerical calculation, in discussing problems connected with simple periodic currents, we need to represent in some manner the phase or direction and maximum magnitude of the current or the electromotive force.

This is most conveniently done by means of complex quantities.

The alternating sine wave is represented in in-

tensity, as well as phase, by a vector $\overline{OI}$, which is determined analytically by two numerical quantities —— the length, $\overline{OI}$, of intensity; and the amplitude, AOI, or phase, θ, of the wave, I.

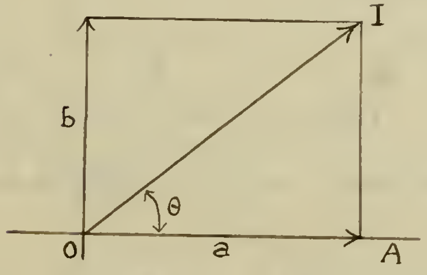

Figure 43.

Instead of denoting the vector which represents the sine wave in the polar diagram by the polar coördinates, I and θ, we can represent it by its rectangular coordinates, a and b (Figure 43) where

$a = I\cos\theta$ is the horizontal component,

$b = I\sin\theta$ is the vertical component of the sine wave.

This representation of the sine wave by its rectangular components is very convenient, in so far as it avoids the use of trigonometric functions in the combination or solution of sine waves.

Since the rectangular components, a and b, are the horizontal and the vertical projections of the vector representing the sine wave, and the projection of the diagonal of a parallelogram is equal to the sum of the projections of its sides, the combination of sine waves by the parallelogram law is reduced to the addition, or subtraction, of their rectangular components. That is:

Sine waves are combined, or resolved, by adding, or subtracting, their rectangular components.

For instance, of a and b are the rectangular components of a sine wave, I, and a^1 and b^1 the components of another sine wave, I^1, (Figure 44) their resultant sine wave, I_o, has the rectangular components a_o equals $(a + a^1)$, and b_o equals $(b + b^1)$.

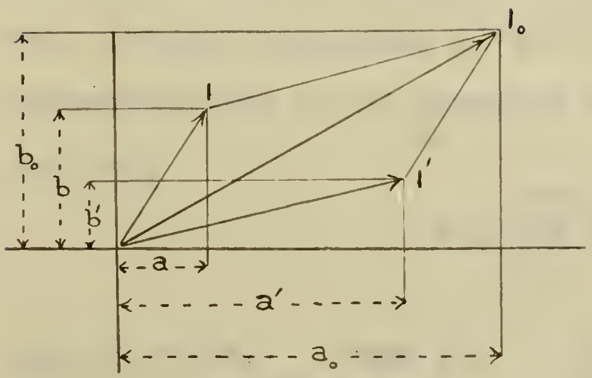

Figure 44.

To get from the rectangular components, a and b, of a sine wave its intensity, i, and phase, θ, we may combine a and b by the parallelogram, and derive

$$i = \sqrt{a^2 + b^2}\,; \qquad \text{(the length or size)}$$

$$\tan\theta = \frac{b}{a} \qquad \text{(the slope).}$$

Hence we can analytically operate with sine waves, as with forces in mechanics, by resolving them into their rectangular components.

If a denotes any line or vector of given length drawn horizontally and to the right, then $-a$ will denote an equal horizontal line to the left. To distinguish, however, the horizontal and the vertical components of sine waves, so as not to be confused in lengthier calculations, we may denote a line of the same length drawn vertically upwards by ja, and a line of the same length drawn verti-

cally downwards by -ja. The otherwise meaningless symbol, j, is therefore an algebraic sign of perpendicularity. We may thus represent the sine wave by the expression

$$I = a + jb,$$

which now has the meaning that a is the horizontal and b the vertical component of the sine wave I, and that both components are to be combined in the resultant wave of intensity,

$$i = \sqrt{a^3 + b^2},$$

and of phase, $\tan \theta = \frac{b}{a}$.

Similarly, a -jb means a sine wave with a as horizontal, and -b as vertical, components. We must remember that the plus sign in the symbol, a + jb, does not imply simple addition, since it connects heterogeneous quantities--horizontal and vertical components--but implies combination by the parallelogram law. For the present, let us think of j as nothing but a distinguishing index.

A wave of equal intensity, and differing in phase from the wave, a + jb, by 180°, or one-half period, is represented in polar coördinates by a vector of opposite direction, and denoted by the expression, -a-jb. Or

> Multiplying the symbolic expression a + jb, of a sine wave by (-1) means reversing the wave, or rotating it through 180°, or one-half period. (1)

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(1) Fleming, J. A. - The Principle of Elementary Wave Telegraphy - Longman's Green Co. page 206.

A wave of equal intensity, but leading a + jb by 90°, or one-quarter period, has (Figure 45) the horizontal component, -b, and the vertical component, a, and is represented symbolically by the expression, ja - b.

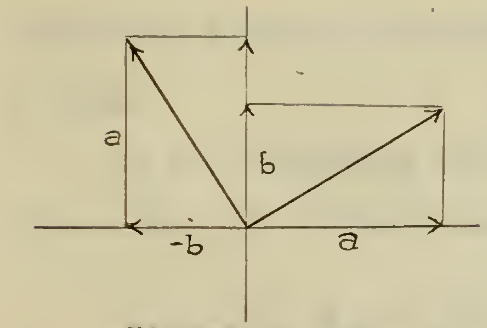

Figure 45.

Multiplying, however, a + jb by j, we get

$$ja + j^2 b;$$

therefore, if we define the heretofore meaningless symbol, j, by the condition,

$$j^2 = -1,$$

we have

$$j (a + jb) = ja - b;$$

hence,

Multiplying the symbolic expression, a + jb, of a sine wave by j means rotating the wave through 90° , or one-quarter period; that is, leading the wave by one-quarter period.

Similarly—

Multiplying by -j means lagging the wave by one-quarter period.

Since

$$j^2 = -1 ,$$

it is

$$j = \sqrt{-1};$$

and

j is the imaginary unit, and the sine wave is represented by a complex imaginary quantity or general number, $a + jb$.

As the imaginary unit, j, has no numerical meaning in the system of ordinary numbers, this definition of

$$j = \sqrt{-1}$$

does not contradict its original introduction as a distinguishing index.

In the vector diagram, the sine wave is represented in intensity as well as phase by one complex quantity,

$$a + jb,$$

where a is the horizontal and b the vertical component of the wave; the intensity is given by

$$i = \sqrt{a^2 + b^2},$$

the phase by

$$\tan \theta = \frac{b}{a},$$

and

$$a = i \cos \theta,$$

$$b = i \sin \theta;$$

hence the wave, a jb, can also be expressed by

$$i (\cos \theta + j \sin \theta),$$

or, by substituting for cos and sin their exponential expressions, we obtain

$$i \varepsilon^{j\theta}.$$

This is the exponential expression of a complex quantity, and the angle θ must be expressed in radians, and not in degrees, that is, with one complete revolution or cycle as 2π, or with $\frac{180}{\pi} = 57.3^\circ$ as a unit.

Sine waves may then be resolved or combined by adding (1) or subtracting their complex algebraic quantities.

For instance, the sine waves

$$a + jb$$

and

$$a^1 + jb^1 ,$$

combined give the sine wave,

$$I = (a + a^1) + j\ (b + b^1).$$

Thus the combination of sine waves is reduced to the elementary algebra of complex quantities.

If $I = i + ji^1$ is a sine wave of alternating current, and r is the resistance, the voltage consumed by the resistance is in phase with the current, and equal to the product of the current and resistance. Or

$$rI = ri + j\ ri^1 .$$

If L is the inductance, and $X = 2\pi fL$ the inductive reactance, the emf produced by the reactance, or counter emf of self-induction, is the product of the current and reactance, and lags in phase 90° behind the current; it is, therefore, represented by the expression

$$-jxL = -jxi + xi^1 .$$

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(1) Steinmetz, C. P. Theory & Calculations of Alternating Current Phenomena - MacGraw Hill Co. pages 34 - 35.

The voltage required to overcome the reactance is consequently 90° ahead of the current and represented by

$$jxI = jxi - xi^1 .$$

Hence, the voltage required to overcome the resistance, r, and the reactance, x, is

$$(r + jx)\ I;$$

that is,

$Z = r + jx$ is the expression of the impedance of the circuit in complex quantities.

Hence, if $I = i + ji^1$ is the current, the voltage required to overcome the impedance, $Z = r + jx$, is

$$E = ZI = (r + jx) \cdot (i + ji^1)$$

$$= (ri + j^2\ xi^1) + j\ (ri^1 + xi);$$

hence, since $j^2 = -1$

$$E = (ri - xi^1) + j\ (ri^1 + xi);$$

or, if $E = e + je^1$ is the impressed voltage and $Z = r + jx$ the impedance, the current through the circuit is

$$I = \frac{E}{Z} = \frac{e + je^1}{r + jx}\ ;$$

or, multiplying both numerator and denominator by (r-jx) to eliminate the imaginary from the denominator, we have

$$I = \frac{(e + je^1)\ (r - jx)}{r^2 + x^2} = \frac{er + je^1r - jex + e^1x}{r^2 + x^2} \qquad (1)$$

$$= \frac{er + e^1x}{r^2 + x^2} + j.\frac{e^1r - ex}{r^2 + x^2}$$

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(1) Steinmetz, C. P. Theory & Calculations of Alternating Current Phenomena - MacGraw Hill Co. Page 37.

Thus, we have the more general form of expressing the sine wave of alternating current by means of complex quantities. This gives us the information necessary to follow the application of complex quantities to the representation of simple periodic quantities.

In representing the maximum value of a simple harmonic electromotive force or current by a vector denoted by such a complex as $a \quad jb$, we fix its position in space because the slope of the vector is such that $\frac{b}{a} = \tan \theta$, and its length or size by $\sqrt{a^2 + b^2}$. If

$$I = i\,(\cos \theta + j \sin \theta)$$

the quantity $(\cos \theta + j \sin \theta)$ is called a rotator, for if applied to any vector it rotates it through an angle θ without changing its size. If we insert the exponential values of sine and cosine, --

$$\sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

and

$$\cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

we have

$$\cos \theta + j \sin \theta = e^{j\theta}$$

Hence $e^{j\theta}$ and $e^{-j\theta}$ are also rotating operators, causing rotation of vectors through an angle θ in the positive or negative direction when applied to them.

If in place of θ we write pt where t signifies time, and $p = \frac{2\pi}{T}$, T being the periodic time, we see that

$$i (\cos pt + j \sin pt) = ie^{jpt}$$

signifies a vector of length i *continually rotating* round one extremity with an angular velocity p. Its real part, namely i cos pt, represents its instantaneous value or projection on a certain axis, and i represents the magnitude or size of its maximum value.

In connection with simple periodic quantities, a (1) theorem of great utility is as follows: If A sin pt represents any simple harmonic quantity, and B cos pt represents another of different amplitude but the same frequency, then A sin pt + B cos pt also represents a simple periodic quantity of amplitude $\sqrt{A^2 + B^2}$, but differing in phase from the first, namely A sin pt, by an angle ϕ, such that

$$\tan \phi = \frac{B}{A}.$$

Hence,

$$A \sin pt + B \cos pt = \sqrt{A^2 + B^2} \sin (pt + \phi).$$

This rule frequently simplifies greatly the derivation of the absolute value and phase angle, from a complicated complex expression.

From this analysis we see that the work with alternating currents no longer requires work in very advanced

(1) Fleming, J. A. The Principles of Elementary Wave Telegraphy - Longman's Green Co. Page 208.

and complicated mathematics but that a simple mastery of the principles and operations of complex numbers will serve as a sufficient background. It is interesting to note that Steinmetz, in 1893, developed this method for making calculations with alternating electrical currents— until that time a sad dilemma to electrical engineers. Fairly overnight he drew the attention of the entire profession of electrical engineering. His solution of the problem came at the psychological moment, when electrical expansion, by means of alternating current, was inevitable, yet woefully retarded because men did not know what Steinmetz had now found out for them.

This intensive work was expounded to electrical engineers, college instructors, college students, and even boys in high school, over a period of years by a masterly series of books, still in use by electrical men throughout the world. They represent the great contribution of Steinmetz to his profession, and through his profession to a world becoming gradually dependent upon the services of electricity.

"A man of peace and good will, yet willing to suffer (1) for his convictions; endowed with marvelous talents in mathematics, and for years a master builder of electrical engineering, Steinmetz, in his comparatively brief life of less than sixty years, contributed imperishably to modern civilization."

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(1) Hammond, J. W. Charles Proteus Steinmetz - The Mentor, May, 1925 - page 3.

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III. CONCLUSION.

CONCLUSION

"Interest is obtained not by thinking about it and consciously aiming at it, but by considering and aiming at the conditions that lie back of it, and compel it. If we can discover a child's urgent needs and powers, and if we can supply an environment of materials, appliances, and resources, physical, social and intellectual--to direct their adequate operation--we shall not have to think about interest. It will take care of itself. The problems of educators, teachers, parents, the state, is to provide the environment that induces educative or developing activities, and where these are found the one thing needful in education is secured." (1) It is for this purpose that these appreciation units have been prepared--to create interest, to unfold certain great ideas, to enrich the experience of the pupil.

Just as we study an epoch in history without expecting to remember all about it, but for the value of the experience, for the frame of mind it gives us and the knowledge of how the human race works out its problems; so we may take our young people through any one of these units of work for the experience it gives them, for the attitudes it creates, and for the knowledge of that side of human experience which we call mathematics.

Through the mathematical experiences encountered in these units, the subject may be made to appeal to the

(1) Dewey, John - Interest and Effort in Education - E. P. Dutton Co. - pages 95 - 96.

learner as interesting and valuable. Furthermore, it has been found that, after this appeal has been successful, pupils need to be held back rather than driven forward in this branch of learning. These units have been selected, not merely for providing opportunity for acquiring more information, but to give the child a chance to grow in habits which are going to be most helpful to him in his own child life as well as in the life of adult society. These topics, if properly developed and enriched, continue with the pupil as interests through life. The child must come to know the contributions of the past to his environment and the possibilities of the future for widening his outlooks and attitudes. The teacher should try to leave each unit with the feeling that there is a fascinating field for further investigation which neither the individual nor the whole group has yet touched.

The mathematics of each unit has been simplified as much as possible,with calculus almost entirely eliminated, in order that the topics may be used to advantage in all high school classes irrespective of their training. The main trend of each unit, however, should be well within the grasp of the whole group and no unit should be carried farther than the group as a whole can go with satisfaction. It is for this reason that these units are written in such detail--in order that the material may be considered as a basis for working out the same or similar units. This is one of the most desirable uses of

these descriptions since the pupils in different classes and environments vary greatly. It is intended also that these units be used by the teacher in order that her preparation in information shall be particularly broad. It is very difficult, almost impossible, for her to gain her information as the unit progresses. She should know the field of subject matter as it spreads out from any unit of work.

The greatest value which any unit of work can have is that it allows for initiative of teacher as well as of pupil. In developing these units, therefore, there has been little thought that they would be taken and used without deviation for any class. The first six of these units may be used with success in both first and second year classes, while the last two, involving some elementary calculus, are better suited for the more advanced pupils. Indeed it is essential that pupils should sense the vigor and vitality of mathematics as early as possible in their high school careers, in order that they may be encouraged to continue their mathematical training at least through high school. For, to quote from the Report of the National Committee on Mathematical Requirements, "if the student who omits the mathematical courses has need of them later, it is almost invariably more difficult, and it is frequently impossible, for him to obtain the training in which he is (1)

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(1) Committee Report - The Reorganization of Secondary Mathematics - The Mathematical Association of America - page 34.

deficient. It requires systematic work under a competent teacher to master properly the technique of the subject, and any break in the continuity of the work is a handicap for which increased maturity rarely compensates. Moreover, when the individual discovers his need of further mathematical training it is usually difficult for him to take the time from his other activities for systematic work in elementary mathematics." We must, then, convince our pupils that mathematics is alive, sparkling, and vigorous. The whole process is to give an intelligible unity to the whole subject, and an attempt to restore that purely intellectual appreciation which has so largely declined during the past generation.

In some classes just a description of the work of a master scientist has proved profitable, while in others the actual working out of the problem itself has been attempted with success. By working out one or two of these units with a class it became apparent that "the world in which we live is incurably mathematical. Our (1) entire civilization, our sciences, our modes of thinking, have a mathematical core." The entire process may cover a wide range. We may convince the youngster who wants to build bridges and skyscrapers that he must mix higher mathematics with his mortar, build it into his foundations; we may reassure the prospective research physician or biologist or chemist or industrial

(1) Betz, Wm. - The Reorganization of Secondary School Mathematics - Progressive Education - page 377.

engineer that he can not reach the top of his profession if he lacks advanced mathematical knowledge.

Mathematics, if we are not greatly mistaken, is presently destined to play a much larger part in our general scheme of education than it ever has in the past. This conclusion is based upon a consideration that the tools and the methods offered by this science have been so largely responsible for the extraordinary advances in other sciences which the past generation has witnessed. The more mathematics contributes to the development of other sciences the more dependent upon it they become. Mathematics is a useful tool, but is also something far greater for it presents in unsullied outline that model after which all scientific thought must be cast. These units, then, are "so many draperies, fashioned to render (1) this outline visible to those who cannot otherwise appreciate it. Even the several branches--analysis, geometry, mechanics--serve the same end; behind them all is the one pure structure of mathematical thought. They who most appreciate the structure will best fashion the draperies, and so render it most clearly visible to those whom they instruct."

On entering into the study of a unit of work, care should be taken to arouse interest. It is just as desirable, however, that sufficient material, making direct appeal to the child's interest, shall be reserved for use

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(1) Carson, G. St.L. - Essays on Mathematical Education - Ginn & Co. - page 31.

in connection with a further study of the topic. Out of his enlarged experiences and interests, therefore, arise increased opportunities for motivating other topics as a means of further natural growth. A deeper appreciation will be evident. A keener observation of all that is going on about him and a more intelligent interpretation of life today will result. In order to accomplish all this, the teacher's own cultural background must be full in its content and significant in its meanings. Facts themselves may be very entertaining, but they are not educative unless they help the child meet his needs and stimulate further creativeness. The teacher's information should be broad and thorough, but it should also have been learned in such a way that it is possible to reassemble it about some element of the child's experience in the classroom in order to make the experience more meaningful to him.

A study of the appreciation units impresses us with the fact that mathematics helps us to understand the world in which we live. The world today is the world of the scientist, the world of the engineer and the mechanic; it is the world of the chemist, the physicist, the biologist, and the physician-scientist; it is equally the world of the astronomer, of the artist, and of the thinker. Man is distinguished from other living creatures by his powers of reasoning; the world in which he lives is a world of thought and of people who think. Mathematics is

essential in any such conception of the world today. Such, in brief, is the appreciation view point.

It may be asked whether it can be hoped that a pupil may end his school career with these ideas fully developed. Frankly, it was not intended that he should. We may regard such ideas as mountain peaks, standing far above the mists of the particular applications which we have considered in these appreciation units. "He who has scrambled longest among the mists sees these peaks most clearly; some indeed have pierced the clouds and seen them in their full beauty--have even scaled them and viewed one from another. The function of the teacher is to lead the child through the mists by such ways as will give him glimpses, even though they are but shadowy, of the higher ground beyond. These will remain and develop in minds to which they are suited--minds, I am convinced, far more common than is generally supposed." (1)

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(1) Carson, G. St.L. - Essays on Mathematical Education - Ginn & Co. - page 139.

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